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Higher Franz-Reidemeister torsion: low dimensional applications

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ABSTRACT. In this expository article I will discuss the theory of higher Franz-Reidemeister torsion and its application to the construction of non-trivial classes in the algebraic K-theory of fields. I will also explain how these ideas may be used to construct classes in the cohomology of the Torelli group.

Introduction

In this note I shall discuss the higher Franz-Reidemeister torsion invariants of [**K**] and [**I-K**$_1$]. What I shall be reporting on is joint work with Kiyoshi Igusa. The first sections will set up the relevent formalism and state the general result. I then provide the results of the known calculations of the invariants in the special case of circle bundles. This leads to a geometric interpretation of the algebraic K-theory of cyclotomic fields. Finally, in the case of surface bundles, I shall show how torsion can be used to construct classes in the rational homology of the Torelli group, and I end by asking whether there is any connection between them and the Miller-Morita-Mumford classes.

I. Preliminaries

In what follows X will denote a fixed topological space, M will be a compact smooth manifold, and $\phi\colon M \to X$ will be a fixed map. We let $\mathit{Diff}(M)$ denote the topological group of diffeomorphisms of M and $\mathit{Maps}(M,X)$ the space of maps from M to X with basepoint ϕ. Then $\mathit{Maps}(M,X)$ has a left $\mathit{Diff}(M)$ action on it defined by $f \cdot \psi = \psi \circ f^{-1}$. Consider the orbit map of ϕ given by

$$\begin{aligned} \mathit{Diff}(M) &\to \mathit{Maps}(M,X) \\ f &\mapsto \phi \circ f^{-1}, \end{aligned}$$

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and let $\mathit{Diff}^{\phi}(M)$ denote the homotopy fibre. This has a delooping, $B\mathit{Diff}^{\phi}(M)$ which can be defined as the balanced product

$$E\mathit{Diff}(M) \times_{\mathit{Diff}(M)} \mathit{Maps}(M, X).$$

We will be mainly interested in the homotopy and homology types of $B\mathit{Diff}^{\phi}(M)$.

Note that $B\mathit{Diff}^{\phi}(M)$ classifies smooth M-bundles $p\colon E \to K$ over a pointed finite complex together with a map $\tilde{\phi}\colon E \to X$ extending $\phi\colon M \to X$ (we are fixing once and for all an identification of M with the fibre over the basepoint of K). This information is summarized in the diagram

$$\begin{array}{lll} M & & \\ \downarrow & \searrow^{\phi} & \\ E & \xrightarrow{\tilde{\phi}} & X \\ \downarrow p & & \\ K. & & \end{array}$$

We will concentrate on two particular cases:

EXAMPLE 1.1. In this situation we take $X = M$ and $\phi\colon M \to M$ the identity. Then $B\mathit{Diff}^{\phi}(M)$ classifies smooth M-bundles which are trivialized as homotopy fibrations. Furthermore,

$$\begin{aligned} B\mathit{Diff}^{\phi}(M) &= G(M)/\mathit{Diff}(M) \\ &= \mathrm{fibre}(B\mathit{Diff}(M) \to BG(M)), \end{aligned}$$

where $G(M)$ is the topological monoid of self homotopy equivalences of M.

EXAMPLE 1.2. $\phi\colon M \to B\pi$ is a map to a $K(\pi, 1)$-space. Let $\tilde{M}$ be the π-covering space of M defined by ϕ. Then $B\mathit{Diff}^{\phi}(M)$ classifies bundles with fibre $\tilde{M}$ = the π-covering space of M defined by ϕ with structure group the π-equivariant diffeomorphisms of $\tilde{M}$.

REMARK 1.3. Regarding these examples, it should be noted that the first fits naturally within the context of the theory of (higher) Whitehead torsion (where we take $p\colon E \to K$ to be a bundle of h-cobordisms which is trivial along one boundary component). The second example is connected with (higher) Franz-Reidemeister torsion, as we shall see below.

We single out a particular case of 1.2 which is relevent to mapping class groups:

EXAMPLE 1.2′. Let M be a closed orientable surface of genus $g > 0$, $\pi = H_1(M)$, and $\phi\colon M \to BH_1(M)$ the classifying map for the universal abelian cover of M. Then $\mathit{Diff}^{\phi}(M)$ consists of diffeomorphisms of M which induce the identity on one-dimensional homology. The set of path components of this is, by definition, the Torelli group T_g. Moreover, the homomorphism of topological groups

$$\mathit{Diff}^{\phi}(M) \xrightarrow{\pi_0} T_g$$

is a homotopy equivalence, i.e. the homotopy groups of $\mathit{Diff}^{\phi}(M)$ in positive dimensions vanish. Consequently, the study of $B\mathit{Diff}^{\phi}(M)$ is the same as the study of BT_g.

II. K-Theory

2.1. Spaces over X. Following Waldhausen [**Wal**], let $R_f(X)$ denote the category of *retractive relative finite CW-complexes over* X. An object of $R_f(X)$ is a space Z endowed with a pair of maps $i\colon X \to Z$, $r\colon Z \to X$ respectively called *inclusion* and *retraction* which satisfy

(1) $r \circ i = \mathrm{id}_X$;
(2) the inclusion i is such that Z is obtained from X by the inductive attachment of a finite number of cells—the attaching order given by the dimension of such cells.

A morphism $Y \to Z$ of $R_f(X)$ is a map of underlying spaces which commutes with inclusion and retraction.

This is a *category with cofibrations* in the sense of Waldhausen. A cofibration $Y \to Z$ is a morphism whose underlying map of spaces is a cofibration in the usual sense of the term.

2.2. Chain complexes. Fix an associative ring R with unit. Let $C_f(R)$ be the category of *finite free chain complexes over* R. An object of $C_f(R)$ is a finite dimensional chain complex which has chains in dimensions ≥ 0 and moreover, which is finitely generated and free in each dimension as an R-module. A morphism of $C_f(R)$ is a chain map. A cofibration is a chain map which is split-injective in each dimension. A weak equivalence is a chain homotopy equivalence. The subcategory of weak equivalences will be denoted by $wC_f(R)$.

Proposition 2.3. *(Waldhausen). The Waldhausen K-theory of $C_f(R)$ is naturally homotopy equivalent to the representing space for Quillen's algebraic K-theory of R, i.e.*

$$\Omega|wS.C_f(R)| \quad \simeq \quad K(R) := \mathbb{Z}_{|n(R)|} \times BGL(R)^+,$$

where on the left, $wS.$ is the $S.$-construction of [**loc. cit.**], *and on the right $n(R)$ is the order of the class defined by R in the Grothendieck group $K_0(R)$, and $^+$ denotes Quillen's plus construction* [**Q**].

III. Linearization Functors

By a *linearization functor*, we shall mean any functor

$$\Lambda\colon R_f(X) \longrightarrow C_f(R)$$

satisfying the following:

(1) Λ is *exact*. By this I mean that Λ preserves the cofibration property, and moreover, if

$$\begin{array}{ccc} A & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & P \end{array}$$

is a pushout diagram in $R_f(X)$ (with cofibrations as horizontal arrows), then Λ applied to the diagram is also a pushout diagram.

(2) $\Lambda(X) = 0$, where X is considered as an object over itself via the identity map, and 0 denotes the trivial chain complex over R.

(3) If Z is an object of $R_f(X)$ and $i\colon X \to Z$ is a homotopy equivalence, then $\Lambda(Z)$ is acyclic.

For a given linearization functor Λ, we obtain a subcategory of weak equivalences in $R_f(X)$ by specifying $Y \to Z$ to be a weak equivalence if and only if $\Lambda(Y) \to \Lambda(Z)$ is a weak equivalence in $C_f(R)$. We let $w_\Lambda R_f(X)$ denote the category of weak equivalences defined by Λ. Then the exactness properties of Λ and 2.3 show that

Proposition 3.1. *Let $A^\Lambda(X) = \Omega|w_\Lambda S.R_f(X)|$ be the Waldhausen K-theory of $R_f(X)$ with respect to the weak equivalences defined by Λ. Then application of Λ defines a map*

$$A^\Lambda(X) \to K(R)$$

which for short will also be denoted by Λ.

Here are a few examples of linearization functors:

Example 3.2. Let G be the fundamental group of X, and define a linearization functor $\Lambda\colon R_f(X) \to C_f(\mathbb{Z}[G])$ by the rule

$$\Lambda(Y) = C_*^{\text{cell}}(Y^G, X^G),$$

where X^G is the universal cover of X, Y^G is the pullback of $X^G \to X$ along the retraction $r\colon Y \to X$, and and $C_*^{\text{cell}}(-)$ denotes the cellular chain complex considered as a free module over $\mathbb{Z}[G]$.

Note that the weak equivalences of $R_f(X)$ with respect to this Λ are the G-equivariant relative homology equivalences, or equivalently by Whitehead's theorem, the relative homotopy equivalences. Consequently, $A^\Lambda(X)$ is identical to Waldhausen's $A(X)$, and the map

$$\Lambda\colon A(X) \to K(\mathbb{Z}[G])$$

of 3.1 is the usual linearization map into algebraic K-theory.

EXAMPLE 3.3. Let $\mathbb{F}$ be a field with involution. Suppose we are provided with a unitary representation $\rho\colon G \to U_r(\mathbb{F})$, where G is the fundamental group of X.

Define a linearization functor $\Lambda(\rho)\colon R_f(X) \to C_f(\mathbb{F})$ by the rule

$$\Lambda(\rho)(Y) = C_*^{\text{cell}}(Y^G, X^G) \otimes_{\mathbb{Z}[G]} \mathbb{F}^r,$$

where on the right hand side the left factor is as in 3.2 and the right factor $\mathbb{F}^r$ has the $\mathbb{Z}[G]$-module structure defined by ρ. Proposition 3.1 then yields a map

$$A^{\Lambda(\rho)}(X) \to K(\mathbb{F}).$$

IV. The Λ-Whitehead Space

Let $A(X)$ be Waldhausen's K-theory of X defined as in 3.2 using the subcategory $hR_f(X)$ of $R_f(X)$ of relative homotopy equivalences.

Let $\Lambda\colon R_f(X) \to C_f(R)$ be an arbitrary linearization functor. By application of axioms (1)-(3) in the definition of a linearization functor, it follows that $hR_f(X) \subset w_\Lambda R_f(X)$. We infer that there is a natural transformation

$$N_\Lambda\colon A(X) \to A^\Lambda(X).$$

As is well-known, there is a natural map

$$i\colon Q_+(X) \to A(X),$$

(which is a split inclusion up to homotopy) where $Q_+(-)$ denotes the unreduced stable homotopy functor. Let $\Omega Wh(X;\Lambda)$ be the homotopy fibre of the composite

$$Q_+(X) \xrightarrow{i} A(X) \xrightarrow{N_\Lambda} A^\Lambda(X) \xrightarrow{\Lambda} K(R).$$

There is then a homotopy fibration

$$\Omega Wh(X;\Lambda) \to Q_+(X) \to K(R).$$

REMARKS 4.1. (1). The notation $\Omega Wh(X;\Lambda)$ for the fibre $Q_+(X) \to K(R)$ indicates that the fibre deloops. This is indeed the case since the latter is an infinite loop map.

(2). If Λ is as in 3.2, then the set of path components of $\Omega Wh(X;\Lambda)$ is canonically isomorphic to the Whitehead group $Wh_1(\pi_1(X))$. If Λ is as in 3.3, the set of path components is canonically isomorphic to the target group for the classical Franz-Reidemeister torsion (see e.g. [**K**]).

V. Higher Torsion

We now return to the context of §1. Namely, we consider classifying spaces associated with data of the form

$$\begin{array}{lll} M & & \\ \downarrow & \searrow^{\phi} & \\ E & \xrightarrow{\tilde\phi} & X \\ \downarrow p & & \\ K & & \end{array}$$

where p is a smooth M-bundle. Fix once and for all a cell structure on M. Form the following object of $R_f(X)$:

$$M_X = M \amalg X,$$

with the obvious inclusion, and where the retraction is $r = \phi \amalg \mathrm{id}_X$. Similarly, let X_X be the object of $R_f(X)$ given by $X \amalg X$ with retraction $\mathrm{id}_X \amalg \mathrm{id}_X$ and with inclusion $X \to X \amalg X$ given by the identity on the second component.

Contractibility Hypothesis 5.1(1). The map $\phi \amalg \mathrm{id}_X \colon M_X \to X_X$ is a weak equivalence in $w_\Lambda R_f(X)$.

Acyclicity Hypothesis 5.1(2). $M_X \in \mathrm{ob} R_f(X)$ is weakly equivalent to the initial object X in $w_\Lambda R_f(X)$, i.e., the R-chain complex $\Lambda(M_X)$ is acyclic.

The main theorem on higher Franz-Reidemeister-Whitehead torsions may be stated as follows:

Theorem 5.2. *(Igusa-Klein* [**K**],[**I-K**$_1$]*). If either the Contractibility Hypothesis 5.1(1) or the Acyclicity Hypothesis 5.1(2) holds, then there is a map*

$$B\mathit{Diff}^{\phi}(M) \xrightarrow{\tau^\Lambda} \Omega Wh(X;\Lambda).$$

Remarks 5.3. (1). If $(p, \tilde\phi)$ are a (fibre bundle, map)-pair as above, then the classifying map followed by the torsion τ^Λ gives rise to a homotopy class in $[K, \Omega Wh(X;\Lambda)]$ which is an invariant of the pair.

(2). The definition of τ^Λ uses the parametrized Morse theory developed by Igusa [**I**$_1$], which in effect says that on any smooth bundle $M \to E \to K$ there exists a "fibrewise generalized framed Morse function" provided that $\dim(K) < \dim(M)$ (this last consideration can be avoided by taking the product of the total space with a disk of sufficiently large dimension). One then shows how to associate to this a "family of poset-filtered chain complexes" parametrized by points of K [**I-K**$_1$]. The torsion is essentially defined in terms of this association.

(3). As stated, 5.2 says nothing about the non-triviality of τ^Λ. However, the calculations of §7 will show that τ^Λ for certain choices of Λ is non-trivial on second homotopy groups.

(4). Suppose X has trivial rational homology (e.g. $X = BG$ for G a finite group). Then $Q_+(X)$ is rationally the same homotopy type as $\mathbb{Z}$ and consequently $\Omega Wh(X;\Lambda)$ has the same rational type as the identity component of $\Omega K(R)$ (assuming that the order of R in its Grothendieck group is infinite). In particular, if $K = S^k$ is a sphere, we get an invariant in $K_{k+1}(R) \otimes \mathbb{Q}$.

The reason for calling τ^Λ "higher torsion" is articulated in the following:

Observation 5.4. *(*[**I-K**$_1$]*). Let Λ be as is in example 3.2. and suppose $\phi\colon M \to X$ is a homotopy equivalence. Then 5.1(1) is satisfied. Regard M as being a bundle over a point (i.e., $M \to M \to *$). Then the path component of $\tau^\Lambda(M,\phi) \in \Omega Wh(X;\Lambda)$ is given by the Whitehead torsion of ϕ (cf. 4.1).*

Similarly, if $\Lambda(\rho)$ as in 3.3 (associated with representation $\rho\colon G \to U_r(\mathbb{F})$), and $\phi\colon M \to BG$, then 5.1(2) is satisfied. Then the path class of $\tau^{\Lambda(\rho)}(M,\phi)$ is the classical Franz-Reidemeister torsion of (M,ρ).

As a further justification for using this terminology, it should be noted that for bundles over S^1 the torsion in our sense coincides with the K_2-type torsion invariants defined by Hatcher-Wagoner [**H-W**] and Wagoner [**W**].

VI. Relation of τ^Λ to the Becker-Gottlieb Transfer

In this section we assume that hypothesis 5.1(2) is satisfied.

Consider the composition

$$BDiff^\phi(M) \xrightarrow{\tau^\Lambda} \Omega Wh(X;\Lambda) \to Q_+(X),$$

where the second map is inclusion of the homotopy fibre. Let us denote this composite by T.

Now suppose we are given a map $f\colon K \to BDiff^\phi(M)$, or equivalently, by §1, a bundle $M \to E \xrightarrow{p} K$ together with extension $\tilde\phi\colon E \to X$ of ϕ. It follows from the construction of τ^Λ that

Proposition 6.1. *T applied to f is homotopic to the Becker-Gottlieb transfer (*[**B-C-G**]*) $p!\colon K \to Q_+(E)$ of p followed by $Q_+(\tilde\phi)\colon Q_+(E) \to Q_+(X)$.*

We remark in passing that there is an analogous result for those bundles which satisfy 5.1(1).

VII. Higher Torsion of Circle Bundles and K_3

The material of this section is summarizes computations carried out in [**I-K**$_2$].

Let $S^1 \to E \xrightarrow{p_n} S^2$ be the circle bundle over the 2-sphere with Euler class n. E is therefore the lens space $L(n,1) = S^3/\mathbb{Z}_n$.

Setting $X = B\mathbb{Z}_n$, and $\phi\colon S^1 \to B\mathbb{Z}_n$ the characteristic map for the n-fold cover $S^1 \xrightarrow{\times n} S^1$, it is clear that ϕ naturally extends to $\tilde\phi\colon E \to B\mathbb{Z}_n$.

Let $\mathbb{Q}(\zeta_n)$ be the cyclotomic number field given by adjoining a primitive n^{th}-root of unity to $\mathbb{Q}$. Let $\rho_n\colon \mathbb{Z}_n \to U_1(\mathbb{Q}(\zeta_n)) = \mathbb{Q}(\zeta_n)^\times$ be the representation defined by sending the generator ζ_n. We then obtain a linearization functor

$$\Lambda(\rho_n)\colon R_f(B\mathbb{Z}_n) \to C_f(\mathbb{Q}(\zeta_n))$$

as in 3.3. Moreover, the Acyclicity Hypothesis (5.1)(2) is clearly satisfied since the homology of the local system on S^1 defined by ρ_n is entirely vanishing. Consequently, the higher Franz-Reidemeister torsion $\tau^\Lambda(p_n)$ is defined and is an element of $\pi_2(\Omega Wh(B\mathbb{Z}_n;\Lambda(\rho)))$ (cf. 5.3(1)).

We next show that this element lifts up to an element of $K_3(\mathbb{Q}(\zeta_n))$.

Let $\partial\colon K_3(\mathbb{Q}(\zeta_n)) \to \pi_2(\Omega Wh(B\mathbb{Z}_n;\Lambda(\rho)))$ be the boundary operator in the homotopy exact sequence of the homotopy fibration

$$\Omega Wh(B\mathbb{Z}_n;\Lambda(\rho)) \to Q_+(B\mathbb{Z}_n) \to K(\mathbb{Q}(\zeta_n)).$$

The following was pointed out to me by Igusa:

Observation. *The Becker-Gottlieb transfer $S^2 \to Q_+(E)$ is null homotopic.*

The reason this is so is simply that the transfer is represented as the Thom construction of a stably framed submanifold of the total space E which in the case at hand is codimension one ($= \dim(S^1)$). Such a manifold must necessarily bound in E in a stably framed way, since $E = L(n,1)$ is parallelizable.

Consequently, by 6.1, the composite $S^2 \xrightarrow{\tau^{\Lambda(\rho)}} \Omega Wh(B\mathbb{Z}_n;\Lambda(\rho)) \to Q_+(B\mathbb{Z}_n)$ is null, and thus the torsion is in the image of the boundary operator ∂.

Let $K_3(\mathbb{Q}(\zeta_n)) \to \mathbb{R}$ be the Borel regulator [**B**].

Theorem 7.1. *(Igusa-Klein [**I-K**$_2$]). Let $x_n \in K_3(\mathbb{Q}(\zeta_n))$ be any lift of $\tau^{\Lambda(\rho)}(p_n)$. Then the value of the Borel regulator on x_n is*

$$n \cdot im(dilog(\zeta_n)),$$

where $dilog(z) = \sum z^k/k^2$ is the dilogorithm function.

In particular, the x_n are distinct for different values of n.

VIII. Higher Milnor Torsion

The idea of this section is that one can sometimes disregard the Contractibility and Acyclicity Hypotheses (5.1(1)(2)) and still obtain higher torsion invariants. A result of this type was announced by Igusa in his 1990 ICM lecture [**I**$_2$].

The idea originally goes back to Milnor [**M**]. In this instance, one has a chain complex C_* over a principal ideal domain R. Milnor constructs a torsion-type invariant which lives in $\mathbb{F}^\times/R^\times$, where $\mathbb{F}$ is the field of fractions of R.

To do this, one appeals to elementary homological algebra to find a chain map $g_*\colon H_*(C)\otimes\mathbb{F} \to C_*\otimes\mathbb{F}$ such that $H_*(C)$ is the homology of C_* considered as a chain complex with zero boundary operators, and such that g_* is the identity on homology.

It follows that g_* is degree-wise injective and so the quotient complex

$$D_* = C_* \otimes \mathbb{F}/g_*(H_*(C) \otimes \mathbb{F})$$

is acyclic. Hence the Reidemeister torsion $\tau(D_*) \in \mathbb{F}^\times/R^\times$ is defined once a basis for D_* is chosen. Such a basis is obtained by choosing a basis for C_* and $H_*(C)$ over R. One then proves that the torsion is independent of the choice of g_*.

We next consider the parametrized case. I will state Igusa's result in the special case $R = \mathbb{Z}$.

THEOREM 8.1. *(Igusa). Let $M \to E \xrightarrow{p} K$ be a smooth manifold bundle such that $H_*(M;\mathbb{Q}) \to H_*(E;\mathbb{Q})$ is injective. Then p determines a torsion invariant*

$$\tau(p) \in [K, \Omega K(\mathbb{Z}) \otimes \mathbb{Q}],$$

where $\otimes\mathbb{Q}$ denotes the rationalization functor from spaces to spaces.

We note that $\tau(p)$ coincides with the torsion τ^Λ of 5.2 when M is a disk.

IX. Torsion and Torelli

Using 8.1, we shall construct classes in $H^{4i}(BT_g)$, where T_g is the Torelli group of a closed surface Σ of genus $g > 0$. Let

$$\Sigma \to BDiff^\phi(\Sigma, *) \to BDiff^\phi(\Sigma)$$

be the universal Σ-bundle with base homotopy equivalent to BT_g (1.2'). As $\Sigma \to BH_1(\Sigma)$ extends to the total space, it is an elementary exercise to see that $H_*(\Sigma) \to H_*(BDiff^\phi(\Sigma,*))$ is injective.

Applying 8.1 together with 5.3(4), we obtain a map

$$BT_g \simeq BDiff^\phi(\Sigma) \longrightarrow \Omega K(\mathbb{Z}) \otimes \mathbb{Q}.$$

Taking cohomology, we get a map

$$H^*(\Omega K(\mathbb{Z});\mathbb{Q}) \to H^*(BT_g;\mathbb{Q}).$$

By Borel [**B**], it is known that the source is a polynomial algebra generated by classes p_i, $i \in \mathbb{N}$, where p_i has dimension $4i$. Pushing these classes forward, we get classes

$$\omega_i \in H^{4i}(BT_g;\mathbb{Q}).$$

On the other hand, we also have the Miller-Morita-Mumford classes

$$\kappa_i \in H^{2i}(BT_g;\mathbb{Q})$$

which are defined by integrating the $(i+1)^{\text{st}}$-power of the Euler class along the fibres of the universal Σ-bundle. It would seem natural to ask

QUESTION 9.1. *Is there a relation between ω_i and κ_{2i}?*

REMARKS 9.2. (1) My original phrasing of 9.1 was somewhat different. I am grateful to Richard Hain for providing the present formulation, which he communicated to me at the time of the workshop.

(2) Added Note: The original conjecture was that the classes in 9.1 should be rational multiples of one another, but recently, Igusa and Penner have shown that there are nontrivial classes y_i arising from K-theory such that $\omega_i - y_i$ is a nonzero multiple of κ_{2i}. The construction of the y_i appeals to Penner's combinatorial description of the moduli space of Riemann surfaces given by "fat graphs".

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