POINCARÉ DUALITY SPACES

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INTRODUCTION

At the end of the last century, Poincaré discovered that the Betti numbers of a closed oriented triangulated topological n-manifold X^n

$$b_i(X) \ := \ \dim_{\mathbb{R}} H_i(X;\mathbb{R})$$

satisfy the relation

$$b_i(X) \ = \ b_{n-i}(X)$$

(see e.g., [Di, pp. 21–22]). In modern language, we would say that there exists a chain map $C^*(X) \to C_{n-*}(X)$ which in every degree induces an isomorphism

$$H^*(X) \cong H_{n-*}(X) \ .$$

The original proof used the dual cell decomposition of the triangulation of X. As algebraic topology developed in the course of the century, it became possible to extend the Poincaré duality theorem to non-triangulable topological manifolds, and also to homology manifolds.

In 1961, Browder [Br1] proved that a finite H-space satisfies Poincaré duality. This result led him to question whether or not every finite H-space has the homotopy type of a closed smooth (= differentiable) manifold. Abstracting further, one asks:

Which finite complexes have the homotopy type of closed topological manifolds? of closed smooth manifolds?

To give these questions more perspective, recall that Milnor had already shown in 1956 that there exist several distinct smooth structures on the 7-sphere [Mi1]. Furthermore, Kervaire [Ke] constructed a 10-dimensional PL manifold with no smooth structure. It is therefore necessary to distinguish between the homotopy types of topological and smooth manifolds. Kervaire and Milnor [K-M] systematically studied groups of the h-cobordism classes of homotopy spheres, where the group structure is induced by connected sum. They showed that these groups are always finite. In dimensions ≥ 5 the h-cobordism equivalence relation is just diffeomorphism, by Smale's h-cobordism theorem [Sm]).

Since topological manifolds satisfy Poincaré duality (with respect to suitable coefficients), the existence of a Poincaré duality isomorphism is a necessary condition for a space to have the homotopy type of a closed manifold. Such a space is called a *Poincaré duality space*, or a *Poincaré complex* for a finite *CW* complex.

Poincaré complexes were to play a crucial role in the Browder-Novikov-Sullivan-Wall surgery theory classification of manifolds. We can view the surgery machine as a kind of *descent theory* for the forgetful functor from manifolds to Poincaré complexes:

- Given a problem involving manifolds, it is often the case that it has an analogue in the Poincaré category.
- One then tries to solve the problem in the Poincaré category, where there is more freedom. In the latter, one has techniques (e.g., homotopy theory) that weren't available to begin with.
- Supposing that there is a solution to the problem in the Poincaré category, the last step is to lift it back to a manifold solution. It is here that the surgery machine applies. Except in low (co)dimensions, the only obstruction to finding the lifting is given by the triviality of a certain element of an L-group $L_n(\pi)$.

Thus surgery theory gives an approach for solving manifold classification problems, *modulo the solution of the corresponding problem for Poincaré complexes.*

In general, a Poincaré duality space is not homotopy equivalent to a topological manifold. Thus Poincaré duality spaces fall into more homotopy types than topological manifolds. In 1965, Gitler and Stasheff [G-S] constructed an example of a simply connected finite complex X which satisfies 5-dimensional Poincaré duality, but which isn't the homotopy type of a closed topological manifold. This example has the homotopy type of a complex of the form $(S^2 \vee S^3) \cup e^5$, with respect to a suitable attaching map $S^4 \to S^2 \vee S^3$. More specifically, X is the total space of a spherical fibration $S^2 \to X \to S^3$ which admits a section. By the clutching construction, such a fibration is classified by an element of $\pi_2(\mathrm{Aut}_*(S^2)) \cong \pi_4(S^2) = \mathbb{Z}/2$. We take X to correspond to the generator of this group.

Returning to Browder's original question about finite H-spaces, it is worth remarking that at the present time there is no known example of a finite H-space which isn't the homotopy type of a closed smooth manifold.

Outline. §1 concerns homology manifolds, which are mentioned more-or-less for their historical interest. In §2 we define Poincaré complexes, following Wall. I then mention the various ways Poincaré complexes can arise. §3 is an ode to the Spivak normal fibration. I give two proofs of its existence. The first essentially follows Spivak, and the second is due to me (probably). In §4 I outline some classification results about Poincaré complexes in low dimensions, and I also give an outline as to what happens in general dimensions in the highly connected case. In §5 I describe some results in Poincaré embedding theory and further connections to embeddings of manifolds. §6 is a (slightly impious) discussion of the Poincaré surgery programs which have been on the market for the last twenty five years or so. I've also included a short appendix on the status of the finite H-space problem. The bibliography has been extended to include related works not mentioned in the text.

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1. Forerunners of Poincaré duality spaces

Spaces having the homological properties of manifolds have a history which dates back to the 1930s, and are to be found in the work of Čech, Lefschetz, Alexandroff, Wilder, Pontryagin, Smith and Begle. These 'generalized n-manifolds' (nowadays called *homology manifolds*) were defined using the local homology structure at a point. The philosophy at the time of their introduction was that these spaces were supposedly easier to work with than smooth or combinatorial manifolds.

We recall the following very special case of the definition (for the general definition and the relevant historical background see [Di, pp 210–213]). An *(ANR) homology n-manifold* X is a compact ANR with local homology groups

$$H_*(X, X \setminus \{x\}) = H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) = \begin{cases} \mathbb{Z} & \text{if } * = n \\ 0 & \text{if } * \neq 0 \end{cases} \quad (x \in X) \ .$$

Now, if our ultimate goal is to study the homotopy properties of manifolds, this definition has an obvious disadvantage: it isn't homotopy invariant. It is easy to construct a homotopy equivalence of spaces $X \xrightarrow{\simeq} Y$ such that X is a homology manifold but Y is not a homology manifold. The notion of Poincaré duality space is homotopy invariant, offering a remedy for the problem by ignoring the local homology structure at each point. Any space homotopy equivalent to a Poincaré duality space is a Poincaré duality space.

2. The definitions

There are several different flavors of Poincaré complex in the literature [Wa3], [Wa4], [Le1], [Spi]. We shall be using Wall's definition in the finite case, without a Whitehead torsion restriction.

Suppose that X is a connected finite CW complex whose fundamental group $\pi = \pi_1(X)$ comes equipped with a homomorphism $w \colon \pi \longrightarrow \{\pm 1\}$, which we shall call an *orientation character*. Let $\Lambda = \mathbb{Z}[\pi]$ denote the integral group ring. Define an involution on Λ by the correspondence $g \mapsto \bar{g}$, where $\bar{g} = w(g)g^{-1}$ for $g \in \pi$. This involution will enable us to convert right modules to left modules and vice-versa. For a right module M, let ${}^wM$ denote the corresponding left module. For a left module N, we let N^w denote the corresponding right module.

Let $C_*(\widetilde{X})$ denote the cellular chain complex of the universal covering space $\widetilde{X}$ of X. Since π acts on $\widetilde{X}$ by means of deck transformations, it follows that $C_*(\widetilde{X})$ is a (finitely generated, free) chain complex of right Λ-modules.

For a right Λ-module M, we may therefore define

$$\begin{aligned} H^*(X; M) &:= H_{-*}(\operatorname{Hom}_\Lambda(C_*(\widetilde{X}), M)) \\ H_*(X; M) &:= H_*(C_*(\widetilde{X}) \otimes_\Lambda {}^wM) \quad . \end{aligned}$$

Given another right Λ-module N, and a class $[X] \in H_n(X; {}^wN)$ we also have a *cap product* homomorphism

$$H^*(X; M) \xrightarrow{\cap[X]} H_{n-*}(X; M \otimes_{\mathbb{Z}} {}^wN)$$

where the tensor product $M \otimes_{\mathbb{Z}} {}^wN$ is given the left Λ-module structure via

$$g \cdot (x \otimes y) := xg^{-1} \otimes gy \quad (g \in \pi, x \otimes y \in M \otimes {}^wN)\ .$$

With respect to these conventions, there is a canonical isomorphism of left modules ${}^w\Lambda \cong \Lambda \otimes_{\mathbb{Z}} {}^w\mathbb{Z}$.

2.1. Definition. The space X is called a *Poincaré complex of formal dimension* n if there is a class $[X] \in H_n(X; {}^w\mathbb{Z})$ such that cap product with it induces an isomorphism

$$\cap[X] \colon H^*(X; \Lambda) \xrightarrow{\cong} H_{n-*}(X; {}^w\Lambda)\ .$$

More generally, a disconnected space X is a Poincaré complex of formal dimension n if each of its connected components is.

We abbreviate the terminology and refer to X as a *Poincaré n-complex.* For the rest of the paper, we shall be implicitly assuming that X is connected. If the orientation character is trivial, we say that X is *orientable*, and a choice of fundamental class $[X]$ in this case is called an *orientation* for X.

2.2. Remark. (1). Wall proved that the definition is equivalent to the assertion that the cap product map

$$H^*(X; M) \xrightarrow{\cap[X]} H_{n-*}(X; {}^wM)$$

is an isomorphism for all left Λ-modules M. In particular, taking $M = \mathbb{Z}$, we obtain the isomorphism $\cap[X] \colon H^*(X, \mathbb{Z}) \cong H_{n-*}(X; {}^w\mathbb{Z})$ as a special case, which amounts to the statement of the classical Poincaré duality isomorphism when w is the trivial orientation character.

(2). Every compact n-manifold X satisfies this form of Poincaré duality.[1] A vector bundle η over S^1 is trivializable if and only if

$$w_1(\eta) = +1 \in H^1(S^1; \mathbb{Z}_2) = \mathbb{Z}_2 = \{\pm 1\}.$$

The homomorphism $w \colon \pi \to \{\pm 1\}$ is defined by mapping a loop $\ell \colon S^1 \to X$ to $+1$ if the pullback of the tangent bundle of X along ℓ is trivializable and -1 otherwise.

(3). In Wall's treatment of surgery theory [Wa4], the above definition of Poincaré complex is extended to include simple homotopy information. This is done as follows: the cap product homomorphism is represented by a chain map $C^*(\widetilde{X}; \Lambda) \to C_{n-*}(\widetilde{X}; {}^w\Lambda)$ of finite degreewise free chain complexes of right Λ-modules. One requires the Whitehead torsion of this chain map to be trivial. In this instance, one says that X is a *simple* Poincaré n-complex. It is known that every compact manifold has the structure of a simple Poincaré complex.

[1] The standard picture of a handle in a manifold, with its core and co-core intersecting in a point, has led Bruce Williams to the following one word proof of Poincaré duality: *BEHOLD!*

2.3. Poincaré pairs. Let (X, A) be a finite CW pair. Assume that X is connected. We assume that X comes equipped with a homomorphism $w\colon \pi_1(X) \to \{\pm 1\}$. We say that (X, A) is a *Poincaré n-pair* if there is a class

$$[X] \in H_n(X, A; {}^w\mathbb{Z})$$

such that cap product with it induces an isomorphism

$$\cap[X]\colon H^*(X;\Lambda) \xrightarrow{\cong} H_{n-*}(X, A; {}^w\Lambda)\,.$$

Moreover, it is required that $\partial_*([X]) \in H_{n-1}(A; {}^w\mathbb{Z})$ equips A with the structure of a Poincaré complex, where the orientation character on A is the one induced by the orientation character on X. Note, however, that in many important examples, A is not connected, even though X is.

2.4. Examples. We mention some ways of building Poincaré complexes.

Gluing. If $(M, \partial M)$ and $(N, \partial N)$ are n-manifolds with boundary or, more generally, Poincaré pairs, and $h\colon \partial M \to \partial N$ is a homotopy equivalence, then the amalgamated union $M \cup_h N$ is a Poincaré n-complex.

A special case of this is the *connected sum* $X \sharp Y$ of two Poincaré complexes X^n and Y^n. To define it, we need to cite a result of Wall: every Poincaré n-complex X has the form $K \cup D^n$, where K is a CW complex and $\dim K < n$; this decomposition is unique up to homotopy (see 4.9 below). Converting the attaching map $S^{n-1} \to K$ into an inclusion $S^{n-1} \subset \bar{K}$, we see that $(\bar{K}, S^{n-1})$ is a Poincaré n-pair. Similarly, with $Y = L \cup D^n$, we may define the connected sum $X \sharp Y$ to be $\bar{K} \cup_{S^{n-1}} \bar{L}$.

Fibrations. Suppose that $F \to E \to B$ is a fibration with F, E and B the the homotopy type of finite complexes. Quinn [Qu2] has asserted that E is a Poincaré complex if and only if F and B are. A proof (using manifold techniques) can be found in a paper of Gottlieb [Got].

This result is important because it explains a wide class of the known examples of Poincaré complexes:

(1) The total space of a spherical fibration over a manifold.
(2) The quotient of a Poincaré complex by a free action of a finite group.

In a somewhat different direction, if a finite group G acts on a Poincaré complex M, then the orbit space M/G satisfies Poincaré duality with rational coefficients. This includes for example the case of orbifolds.

S-duality. Let K and C be based spaces, and suppose that

$$d\colon S^{n-2} \to K \wedge C$$

is an *S-duality map*, meaning that slant product with the homology class $d_*([S^{n-2}]) \in \widetilde{H}_{n-2}(K \wedge C)$ induces an isomorphism in all degrees $f : \widetilde{H}^*(K) \cong \widetilde{H}_{n-*-2}(C)$.

Let $P\colon \Sigma(K \wedge C) \to \Sigma K \vee \Sigma C$ denote the *generalized Whitehead product* map, whose adjoint $K \wedge C \to \Omega\Sigma(K \vee C)$ is defined by taking the loop commutator $[i_K, i_C]$ (Samelson product), where $i_K\colon K \to \Omega\Sigma(K \vee C)$ and $i_C\colon C \to \Omega\Sigma(K \vee C)$ are adjoint to the inclusions (see [B-S, p. 192]).

The CW complex

$$X := (\Sigma K \vee \Sigma C) \cup_{P \circ \Sigma d} D^n$$

is a Poincaré n-complex, with

$$\begin{aligned} \cap[X] = \begin{pmatrix} 0 & \pm f^* \\ f & 0 \end{pmatrix} : H^{n-*}(X) \ &= \ \widetilde{H}^{n-*-1}(K) \oplus \widetilde{H}^{n-*-1}(C) \\ \cong H_*(X) \ &= \ \widetilde{H}_{*-1}(K) \oplus \widetilde{H}_{*-1}(C) \quad (* \neq 0, n) \ . \end{aligned}$$

(The proof uses [loc. cit., 4.6, 5.14]; see also 4.10 below). Spaces of this kind arise in higher dimensional knot theory, where X is the boundary of a tubular neighborhood of a Seifert surface $V^n \subset S^{n+1}$ (i.e., the double $V \cup_{\partial V} V$ of $(V, \partial V)$) of a knot $S^{n-1} \subset S^{n+1}$.

Given X^n as above, we can form a Poincaré $(n+2)$-complex Y^{n+2} by applying the same construction to the doubly suspended S-duality $\Sigma^2 d\colon S^n \to \Sigma K \wedge \Sigma C$. Thus iterated application of the operation

$$(K, C, d) \quad \mapsto \quad (\Sigma K, \Sigma C, \Sigma^2 d)$$

gives rise to a *periodic family* of Poincaré complexes. This type of phenomenon is related to the periodicity of the high-dimensional knot cobordism groups.

3. The Spivak fibration

A compact smooth manifold M^n comes equipped with a tangent bundle τ_M, whose fibres are n-dimensional vector spaces. Embedding M in a high dimensional euclidean space $\mathbb{R}^{n+k}$, we can define the *stable normal bundle* ν, which is characterized by the equation

$$\tau_M \oplus \nu_M = 0$$

in the reduced Grothendieck group of stable vector bundles over M. By identifying a closed tubular neighborhood of M^n in $\mathbb{R}^{n+k}$ with the normal disk bundle $D(\nu)$, and collapsing its complement to a point (the *Thom-Pontryagin construction*), we obtain the *normal invariant*[2]

$$\alpha\colon S^{n+k} = (\mathbb{R}^{n+k})^+ \xrightarrow{\text{collapse}} \mathbb{R}^{n+k}/(\mathbb{R}^{n+k} - \mathrm{int} D(\nu_M)) \xrightarrow[\cong]{\text{excision}} T(\nu_M),$$

in which $T(\nu) = D(\nu_M)/S(\nu_M)$ is the *Thom space* of ν (here, $S(\nu_M)$ denotes the normal sphere bundle of ν_M). The map α satisfies

$$U \cap \alpha_*([S^{n+k}]) = [M]\,,$$

[2] The use of this term in the literature tends to vary; here we have chosen to follow Williams [Wi1].

where $U \in H^k(D(\nu_M), S(\nu_M); \mathbb{Z}^t)$ denotes a Thom class for ν_M, in which the latter cohomology group is taken with respect to the local coefficient system defined by the first Stiefel-Whitney class of ν_M (i.e., the orientation character of M).

The above relation between the normal invariant, the Thom class and the fundamental class is reflected in an observation made by Atiyah. If $p\colon D(\nu_M) \to M$ denotes the bundle projection, then the assignment $v \mapsto (v, p(v))$ defines a map of pairs

$$(D(\nu_M), S(\nu_M)) \longrightarrow (D(\nu_M) \times M, S(\nu_M) \times M)$$

which induces a map of associated quotients

$$T(\nu_M) \longrightarrow T(\nu_M) \wedge M_+ \,,$$

where M_+ denotes M with the addition of a disjoint basepoint. Composing this map with the normal invariant, we obtain a map

$$S^{n+k} \xrightarrow{d} T(\nu_M) \wedge M_+ \,.$$

3.1. Theorem. *(Atiyah Duality* [At]*). The map d is a Spanier-Whitehead duality map, i.e., slant product with the class $d_*([S^{n+k}]) \in \widetilde{H}_{n+k}(T(\nu_M) \wedge M_+)$ yields an isomorphism*

$$\widetilde{H}^*(T(\nu_M)) \quad \cong \quad \widetilde{H}_{n+k-*}(M_+) \,.$$

With respect to this isomorphism (or rather, taking a version of it with twisted coefficients), we see that a Thom class U maps to a fundamental class $[M]$ and the map is given by cap product with $\alpha_*([S^{n+k}])$. Thus, the relation $U \cap \alpha_*([S^{n+k}]) = [M]$ is a manifestation of the statement that the Thom complex $T(\nu_M)$ is a Spanier-Whitehead dual of M_+.

The above discussion was intended to motivate the following:

3.2. Definition. Let X be a Poincaré n-complex with orientation character w. By a *Spivak normal fibration* for X, we mean

- a $(k-1)$-spherical fibration $p\colon E \to X$, and
- a map
$$S^{n+k} \xrightarrow{\alpha} T(p) \,,$$
where $T(p) = X \cup CE$ denotes the mapping cone of p.

Moreover, we require that

$$U \cap \alpha_*([S^{n+k}]) = [X] \,,$$

where $U \in H^k(p; \mathbb{Z}^w)$ is a Thom class for the spherical fibration p (here we are taking the cohomology group of the pair $(X \cup_p E \times I, E \times 0)$ defined by the mapping cylinder of p and the coefficients are given by the local system on X defined by the orientation character w).

The map $\alpha\colon S^{n+k} \to T(p)$ is called a *normal invariant.*

3.3. Theorem. *(Spivak). Every Poincaré n-complex X admits a Spivak normal fibration with fibre S^{k-1}, provided that $k \gg n$. Moreover, it is unique in the following sense: given two Spivak fibrations (E_0, p_0, α_0) and (E_1, p_1, α_1) with respect to the same integer k, then there exists a stable fibre homotopy equivalence*

$$h\colon E_1 \xrightarrow{\simeq} E_2$$

such that the induced map $T(h)\colon T(p_0) \to T(p_1)$ composed with α_0 is homotopic to α_1.

Actually, Spivak only proves this in the 1-connected case, but a little care shows how to extend to result to the non-simply connected case.

Let me now give Spivak's construction. As X is a finite complex, we can identify it up to homotopy with a closed regular neighborhood N of a finite polyhedron in euclidean space $\mathbb{R}^{n+k}$. Let $p\colon E \to X$ be the result of converting the composite

$$\partial N \to N \simeq X$$

into a fibration. One now argues that the homotopy fibre of p is homotopy equivalent to a $(k-1)$-sphere. To see this, we combine n-dimensional Poincaré duality for X together with the $(n+k)$-dimensional Poincaré duality for $(N, \partial N)$ (the latter having trivial orientation character) to conclude that

$$\begin{aligned} H^*(X;\Lambda) &\cong H_{n-*}(X; {}^w\Lambda) \\ &\cong H_{n-*}(N; {}^w\Lambda) \\ &\cong H^{k+*}(N, \partial N; ({}^w\Lambda)^e) \\ &\cong H^{k+*}(p; ({}^w\Lambda)^e)\,, \end{aligned}$$

where $({}^w\Lambda)^e$ denotes the effect of converting ${}^w\Lambda$ to a right module by means of the trivial orientation character $e(g) := 1$.

Now, it is straightforward to check that this isomorphism is induced by cup product with a class $U \in H^k(p; \mathbb{Z}^w)$, so it follows that the fibration $p\colon E \to X$ satisfies the Thom isomorphism with respect to twisted coefficients. However, by the following, such fibrations are spherical fibrations.

3.4. Lemma. *(Spivak* [Spi, 4.4]*, Browder* [Br4, I.4.3]*). Suppose that $p\colon E \to B$ is a fibration of connected spaces whose fibre F is 1-connected. Then $F \simeq S^{k-1}$, $k \geq 2$, if and only if the generalized Thom isomorphism holds, i.e., there exists a class $U \in H^k(p; \mathbb{Z}^w)$ (with respect to some choice of orientation character $w\colon \pi_1(B) \to \{\pm 1\}$) such that cup product induces an isomorphism*

$$U\cup\colon H^*(B;\Lambda) \to H^{*+k}(p; ({}^w\Lambda)^e) \quad .$$

(The original proof of this lemma involves an intricate argument with spectral sequences. For an alternative, non-computational proof see Klein [Kl1].)

To complete the proof of the existence of the normal fibration, we need to construct a normal invariant $\alpha\colon S^{n+k} \to T(p)$. By definition, $T(p)$ is homotopy equivalent to $N/\partial N$, nd the latter comes equipped with a degree one map

$$S^{n+k} \to N/\partial N$$

given by collapsing the exterior of N to a point. This defines α.

Observe that when X is a smooth manifold then the Spivak fibration $E \longrightarrow X$ admits a reduction to a k-plane bundle with structure group $O(k)$, i.e., the stable normal bundle of X. Similar remarks apply to PL and topological manifolds. This observation gives the first order obstruction to a finding a closed (TOP, PL or DIFF) manifold which is homotopy equivalent to a given Poincaré complex: *the normal fibration should admit a* (TOP, PL *or* DIFF) *reduction.*

We wish to illustrate the utility of this by citing a result from surgery theory.

3.5. Theorem. *(Browder, cf.* [Ra4, p. 210]*). If X is a 1-connected Poincaré complex of dimension ≥ 5, then X is homotopy equivalent to a closed topological manifold if and only if the normal fibration for X admits a TOP-reduction.*

As a corollary, we see that every finite 1-connected H-space of dimension ≥ 5 is homotopy equivalent to a topological manifold: the Spivak fibration in this case is trivializable (cf. Browder and Spanier [Br-Sp]), so we may take the trivial reduction.

3.6. An alternative approach. The above construction of the Spivak normal fibration required us to identify the Poincaré complex X with a regular neighborhood of a finite polyhedron in $\mathbb{R}^n$. From an aesthetic point of view, it is desirable to have a construction which altogether avoids the theory of regular neighborhoods. The following, which was discovered by the author, achieves this. To simplify the exposition, we shall only consider the case when $\pi_1(X)$ is trivial, and leave it to the reader to fill-in the details in the general case.

Let G be a topological group (which to avoid pathology, we assume is a CW complex). Consider based G-spaces built up inductively from a point by attaching free G-cells $D^j \wedge G_+$ along their boundaries $S^{j-1} \wedge G_+$. Such G-spaces are the free, based G-CW complexes. We shall call such G-spaces *cofibrant*.

Given a cofibrant G-space Y, define the *equivariant cohomology* of Y by

$$\widetilde{H}^*_G(Y) := \widetilde{H}^*(Y/G; \mathbb{Z})$$

where the groups on the right are given by taking reduced singular cohomology.

Similarly, we have the *equivariant homology* of Y

$$\widetilde{H}^G_*(Y) := \widetilde{H}_*(Y/G; \mathbb{Z})\,.$$

Given two G-spaces Y and Z, we can form their smash product $Y \wedge Z$. Gives this the diagonal G-action, and let $Y \wedge_G Z$ denote the resulting orbit space.

3.7. Definition. Assume that $\pi_0(G)$ is trivial. A map of based spaces $d\colon S^m \to Y \wedge_G Z$ is said to be an *equivariant duality map* if the correspondence $x \mapsto x/d_*([S^m])$ defines an isomorphism

$$\widetilde{H}^*_G(Y) \xrightarrow{\cong} \widetilde{H}^G_{m-*}(Z)\,.$$

3.8. Remarks. (1). Another way of saying this is that the evident composite

$$S^m \to Y \wedge_G Z \to (Y/G) \wedge (Z/G)$$

is an S-duality map.

(2). Our definition is a dual variation of one given by Vogell [Vo], and the set-up is similar to Ranicki [Ra1, §3] who defines an analogue for discrete groups. If G is not connected, then the definition is slightly more technical in that we have to take cohomology with $\Lambda = \mathbb{Z}[\pi_0(G)]$-coefficients.

Now, using a cell-by-cell induction (basically, Spanier's exercises [Spa, pp. 462–463] made equivariant), one verifies that every finite cofibrant G-space Y (i.e., which is built up from a point by a finite number of G-cells) has the property that there exists a finite G-space Z and an equivariant duality map $S^m \to Y \wedge_G Z$ for some choice of $m \gg 0$.

It is well-known that any connected based CW complex X comes equipped with a homotopy equivalence $BG \xrightarrow{\simeq} X$, where G is a suitable topological group model for the loop space of X (e.g., take G to be the geometric realization of the underlying simplicial set of the Kan loop group of the total singular complex of X). Here, BG denotes the classifying space of X. Let EG be the total space of a universal bundle over X. Then EG is a free contractible G-space. Let EG_+ be the effect of adjoining a basepoint to EG. Since BG is homotopy finite, it follows that EG_+ is the equivariant type of a finite cofibrant G-space. Hence, there exists an equivariant duality map

$$S^m \xrightarrow{d} EG_+ \wedge_G Z := Z_{hG}$$

for suitably large m, where $Z_{hG} := (EG \times_G Z)/(EG \times_G *)$ is the reduced Borel construction of G acting on Z (note in fact that Z_{hG} is homotopy equivalent to Z/G since Z is assumed to be cofibrant).

In what follows, we assume that $m \gg n =: \dim X$.

3.9. Claim. *If BG has the structure of an n-dimensional Poincaré complex, then Z is unequivariantly homotopy equivalent to a sphere of dimension $m-n-1$.*

Proof. Combining Poincaré duality with equivariant duality, we obtain an isomorphism

$$\widetilde{H}_{m-n+*}(Z_{hG}) \cong \widetilde{H}^{n-*}(BG_+) \cong \widetilde{H}_*(BG_+)\,.$$

One checks that this isomorphism is induced by cap product with a suitable class $U \in \widetilde{H}^{m-n}(Z_{hG})$. Now observe that up to a suspension, Z_{hG} is the mapping cone of the evident map

$$EG \times_G Z \to BG$$

and it follows that Z_{hG} amounts to the Thom complex for this map converted into a fibration. It follows that the Thom isomorphism is satisfied, and we conclude by 3.4 above that its fibre Z has the homotopy type of an $(m-n-1)$-sphere.

To complete our alternative construction of the Spivak fibration, we need to specify a normal invariant α. This is given by the duality map $d\colon S^m \to Z_{hG}$.

4. The classification of Poincaré complexes

We outline the classification theory of Poincaré complexes in two instances: (i) low dimensions, and (ii) the highly connected case. In (i), we shall see that the main invariants are of Postnikov and tangential type, and ones derived from them. In (ii), the Hopf invariant is the main tool.

4.1. Dimension 2. Every orientable Poincaré 2-complex is a homotopy equivalent to a closed surface (see Eckmann-Linnell [E-L] and Eckmann-Müller [E-M]). Surprisingly, this is a somewhat recent result.

4.2. Dimension 3. Clearly, Poincaré duality implies that a 1-connected Poincaré 3-complex X is necessarily homotopy equivalent to S^3.

Wall [Wa3] studied Poincaré 3-complexes X in terms of the fundamental group $\pi = \pi_1(X)$, the number of ends e of π and the second homotopy group $G = \pi_2(X)$. The condition that $e = 0$ is the same as requiring π to be finite. It follows that the universal cover of X is homotopy equivalent to S^3, so G is trivial in this instance.

It turns out in this case that π is a group *period* 4, meaning that $\mathbb{Z}$ admits a periodic projective resolution of $\mathbb{Z}[\pi]$ modules of period length 4. Wall showed that the first k-invariant of X is a generator g of $H^4(\pi;\mathbb{Z})$ (the latter which is a group of order $|\pi|$). The assignment $X \mapsto (\pi_1(X), g)$ was proved to induce a bijection between the set of homotopy types of Poincaré complexes and the the set of pairs (π, g) with π finite of period 4 and $g \in H^4(\pi;\mathbb{Z})$ a generator, modulo the equivalence relation given by identifying (π, g) with (π', g') if there exists an isomorphism $\pi \longrightarrow \pi'$ whose induced map on cohomology maps g' to g.

In the case when $e \neq 0$, then π is infinite and $\widetilde{X}$ is non-compact. If $e = 1$, homological algebra shows that $\widetilde{X}$ is contractible in this case, so X is a $K(\pi, 1)$.

If $e = 2$, the Wall shows that X is homotopy equivalent to one of $\mathbb{RP}^3 \sharp \mathbb{RP}^3$, $S^1 \times \mathbb{RP}^2$ or the one of the two possible S^2-bundles over S^1. This summarizes the classification results of Wall for groups for π in which $e \leq 2$.

In 1977, Hendriks [He] showed that the homotopy type of a connected Poincaré 3-complex X is completely determined by three invariants:

- the fundamental group $\pi = \pi_1(X)$,
- the orientation character $w \in \mathrm{Hom}(\pi, \mathbb{Z}/2)$, and
- the element $\tau := u_*([X]) \in H_3(B\pi; {}^w\mathbb{Z})$ given by taking the image of the fundamental class with respect to the homomorphism $H_3(X; {}^w\mathbb{Z}) \longrightarrow H_3(B\pi_1(X); {}^w\mathbb{Z})$ induced by the classifying map $u : X \longrightarrow B\pi$ for the universal cover of X.

Call such data a *Hendriks triple*.

Shortly thereafter, Turaev [Tu] characterized those Hendriks triples (π, w, τ) which are realized by Poincaré complexes, thereby completing the classification. For a ring Λ, let ho-$\mathbf{mod}_\Lambda$ be the category of fractions associated to the category of right Λ-modules given by formally inverting the class of morphisms $0 \to P$, where P varies over the finitely generated projective modules. Call a homomorphism $M \to N$ of right Λ-modules a *P-isomorphism* if it induces an isomorphism in ho-$\mathbf{mod}_\Lambda$.

Set $\Lambda = \mathbb{Z}[\pi]$, where π is a finitely presented group which comes equipped with an orientation character $w\colon \pi \to \{\pm 1\}$. Let $I \subset \Lambda$ denote the augmentation ideal, given by taking the kernel of the ring map $\Lambda \to \mathbb{Z}$ defined on group elements by $g \mapsto 1$. In particular, I is right Λ-module.

Choose a free right Λ-resolution

$$\cdots \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \to I \to 0$$

of I, with C_1 and C_2 finitely generated. Let $C^* := \hom_\Lambda(C_i, \Lambda)$ denote the corresponding complex of dual (left) modules. Let J be the right module given by taking the cokernel of the map

$$(C_1^*)^w \xrightarrow{(d_2^*)^w} (C_2^*)^w\,.$$

Then Turaev shows that there is an isomorphism of abelian groups

$$A\colon \hom_{\text{ho-}\mathbf{mod}_\Lambda}(J, I) \quad \xrightarrow{\cong} \quad H_3(\pi; {}^w\mathbb{Z})\,.$$

4.3. Theorem. *(Turaev). A Hendriks triple $x := (\pi, w, \tau)$ is realized by a Poincaré 3-complex if in only if $\tau = A(t)$ for some P-isomorphism $t\colon J \to I$.*

4.4. Dimension 4. Milnor [Mi2] proved that the intersection form

$$H_2(X^4) \otimes H_2(X^4) \to \mathbb{Z}$$

(or equivalently, the cup product pairing on 2-dimensional cohomology) determines the homotopy type of a simply connected Poincaré 4-complex, and that every unimodular symmetric bilinear form over $\mathbb{Z}$ is realizable. We should perhaps also mention here the much deeper theorem of Freedman, which says that the *homeomorphism* type of a closed topological 4-manifold is determined by its intersection form and its Kirby-Siebenmann invariant (the latter is a $\mathbb{Z}/2$-valued obstruction to triangulation).

We may therefore move on to the non-simply connected case. It is well-known that any group is realizable as the fundamental group of a closed 4-manifold, and hence of a Poincaré 4-complex. Given a Poincaré 4-complex X with fundamental group π, the obvious invariants which come to mind are $G := \pi_2(X)$ and the intersection form on the universal cover, which can be rewritten as $\phi\colon G \times G \to \mathbb{Z}$ (since $\pi_2(X) = H_2(\widetilde{X})$); the group π acts via isometries on the latter.

Wall [Wa3] studied oriented Poincaré 4-complexes X^4 whose fundamental group is a cyclic group of prime order $p \neq 2$. Wall showed under these assumptions that the homotopy type of X is determined by G and the intersection form $G \times G \to \mathbb{Z}$. However, when π is the group of order 2, this intersection form is too weak to detect the homotopy type of X (see [H-K, 4.5]).

Hambleton and Kreck [H-K] extended Wall's work to the case when π is a finite group with periodic cohomology of order 4. To a given oriented X^4, they associate a 4-tuple

$$(\pi, G, \phi, k)$$

where $\pi = \pi_1(M)$, $G = \pi_2(M)$, $\phi\colon G \times G \longrightarrow \mathbb{Z}$ denotes the intersection form and $k \in H^3(\pi; G)$ denotes the first Postnikov invariant of X. Such a system is called the *quadratic* 2*-type* of X. Moreover generally, one can consider all such 4-tuples, and define *isometry* $(\pi, G, \phi, k) \longrightarrow (\pi', G', \phi', k')$ consist of isomorphisms $\pi \cong \pi'$ and $G \cong G'$ which map ϕ to ϕ' and k to k'.

4.5. Theorem. *(Hambleton-Kreck). Let X^4 be a closed oriented Poincaré complex with $\pi = \pi_1(X)$ a finite group having* 4*-periodic cohomology. Then the homotopy type of X is detected by the isometry class of its quadratic* 2*-type.*

Notice that the result fails to identify the possible quadratic 2-types which occur for Poincaré complexes. Bauer [Bauer] extended this to finite groups π whose Sylow subgroups are 4-periodic. Teichner [Te] extended it to the non-orientable case where a certain additional secondary obstruction appears. Teichner also realizes the obstruction by exhibiting a non-orientable Poincaré 4-complex having the same quadratic 2-type as $\mathbb{RP}^4 \sharp \mathbb{CP}^2$, but the two spaces have different homotopy types. Thus Teichner's secondary obstruction may be non-trivial. Other examples in the non-orientable case were constructed by Ho, Kojima, and Raymond [H-K-R].

Another approach to classification in dimension 4 is to be found in the works of Hillman (see e.g., [Hill]).

We should also mention here the work of Baues [Baues] which a provides a (rather unwieldy but) complete set of algebraic invariants for all 4-dimensional CW complexes.

4.6. Dimension 5**.** The main results in this dimension assume that the fundamental group is trivial. Madsen and Milgram [M-M, 2.8] determined all Poincaré 5-complexes with 4-skeleton homotopy equivalent to $S^2 \vee S^3$. They show that such a space is homotopy equivalent to one of the following:

(1) $S^2 \times S^3$,
(2) $S(\eta \oplus \epsilon^2)$ = the total space of the spherical fibration that is given by taking the fibrewise join of the Hopf fibration $S^3 \xrightarrow{\eta} S^2$ with the trivial fibration $\epsilon^2\colon S^2 \times S^1 \longrightarrow S^2$, or
(3) the space given by attaching a 5-cell to $S^2 \vee S^3$ by means of the map $S^4 \longrightarrow S^2 \vee S^3$ given by $[\iota_2, \iota_3] + \eta^2\iota_2$, where $[\iota_2, \iota_3]\colon S^4 \longrightarrow S^2 \vee S^3$ denotes the attaching map for the top cell of the cartesian product $S^2 \times S^3$ (= the Whitehead product), $\eta^2\colon S^4 \longrightarrow S^2$ denotes the composite $\Sigma\eta\colon S^4 \longrightarrow S^3$ followed by η, and $\iota_2\colon S^2 \rightarrow S^2 \vee S^3$ denotes the inclusion.

The last of these cases is the Gitler-Stasheff example mentioned in the introduction, and hence fails to have the homotopy type of a closed smooth 5-manifold. This can be seen by showing that the Thom space of the associated Spivak normal bundle fails to be the Thom space of a smooth vector bundle.

Stöcker has completely classified 1-connected Poincaré 5-complexes up to oriented homotopy type. To a given oriented X^5, we may associate the system of invariants

$$I(X) \quad := \quad (G, b, w_2, e)$$

where

- $G = H_2(X)$,

- $b\colon T(G)\times T(G)\longrightarrow \mathbb{Q}/\mathbb{Z}$ is the linking form for the torsion subgroup $T(G)\subset G$,
- $w_2\in \operatorname{Hom}(G,\mathbb{Z}/2)$ is the second Stiefel-Whitney class for the Spivak fibration of X (which makes sense since $\operatorname{Hom}(\pi_2(BSG),\mathbb{Z}/2)=\mathbb{Z}/2$, where the space BSG classifies oriented stable spherical fibrations), and
- $e\in H^3(X;\mathbb{Z}/2)\cong G\otimes \mathbb{Z}/2$ denotes the obstruction linearizing the Spivak-fibration over the 3-skeleton of X (we are using here that the map $BSO\to BSG$ is 2-connected, so a linearization always exists over the 2-skeleton).

We remark that the first three of these invariants was used by Barden [Bar] to classify 1-connected smooth 5-manifolds.

More generally, one can consider tuples (G,b,w_2,e) in which G is a finitely generated abelian group, $b\colon T(G)\times T(G)\longrightarrow \mathbb{Q}/\mathbb{Z}$ is a nonsingular skew symmetric form, $w_2\colon G\longrightarrow \mathbb{Z}/2$ is a homomorphism and $e\in G\otimes\mathbb{Z}/2$ is an element. The data are required to satisfy $w_2(x)=b(x,x)$ for all $x\in T(G)$ and $(w_2\otimes \mathrm{id})(e)=0$. It is straightforward to define isomorphism and direct sums of these data, so we may define J to be the semi-group of isomorphism classes of such tuples.

4.7. Theorem. *(Stöcker* [Sto]*). The assignment* $X^5\mapsto I(X^5)$ *defines an isomorphism between* J *and the semigroup of oriented homotopy types of* 1*-connected Poincaré* 5*-complexes, where addition in the latter is defined by connected sum.*

Using a slightly different version of this, it is possible to write down a complete list of oriented homotopy types of 1-connected Poincaré 5-complexes in terms of 'atomic' ones and the connected sum operation (see [loc. cit., 10.1]).

4.8. The highly connected case. In "Poincaré Complexes: I", Wall announces that the classification of 'highly connected' Poincaré complexes will appear in the forthcoming part II. Unfortunately, part II never did appear. We shall recall some of the homotopy theory which would presumably enter into a hypothetical classification in the metastable range.

To begin with, it is well-known that a closed n-manifold can be given the structure of a finite n-dimensional CW complex with one n-cell. The analogue of this for Poincaré complexes was proved by Wall [Wa3, 2.4], [Wa4, 2.9] and is called the *disk theorem*:

4.9. Theorem. *(Wall). Let* X *be a finite Poincaré* n*-complex. Then* X *is homotopy equivalent to a* CW *complex of the form* $L\cup_\alpha D^n$*. If* $n\neq 3$ *then* L *can be chosen as a complex with* $\dim L\le n-1$ *(when* $n=3$*,* L *can be chosen as finitely dominated by a* 2*-complex). Moreover, the pair* (L,α) *is unique up to homotopy and orientation.*

Suppose that X is a n-dimensional CW complex of the form $(\Sigma K)\cup_\alpha D^n$, with K connected. We want to determine which attaching maps $\alpha\colon S^{n-1}\longrightarrow \Sigma K$ give X the structure of a Poincaré complex. To this end, we recall the James-Hopf invariant

$$\pi_{n-1}(\Sigma K)\xrightarrow{H}\pi_{n-1}(\Sigma K\wedge K)$$

which is defined using the using the well-known homotopy equivalence $J(K) \xrightarrow{\simeq} \Omega\Sigma K$, where $J(K)$ denotes the free monoid on the points of K. In terms of this identification, H is induced by the map $J(K) \to J(K \wedge K)$ given by mapping a word $\prod_i x_i$ to the word $\prod_{i<j} x_i \wedge x_j$.

4.10. Theorem. *(Boardman-Steer* [B-S, 5.14]*). Up to homotopy, the reduced diagonal $\Delta\colon X \to X \wedge X$ factors as*

$$X \xrightarrow{\text{pinch}} S^n \xrightarrow{\Sigma H(\alpha)} \Sigma K \wedge \Sigma K \xrightarrow{\subset} X \wedge X\,,$$

where the first map in this factorization is given by collapsing $\Sigma K \subset X$ to a point.

Since the slant product is induced by the reduced diagonal, we obtain,

4.11. Corollary. *A map $\alpha\colon S^{n-1} \to \Sigma K$ gives rise to a Poincaré n-complex $X = (\Sigma K) \cup_\alpha D^n$ if and only if its Hopf invariant*

$$H(\alpha)\colon S^{n-1} \to \Sigma K \wedge K$$

is a Spanier-Whitehead duality.

In particular, this result says that the complex K is self-dual whenever X is a Poincaré complex (compare [Wa1, 3.8]).

Suppose now that we are given CW complex $X = L \cup_\alpha D^n$ which $(r-1)$-connected. If X is to be a Poincaré complex, then it would follow by duality that L is homotopy equivalent to a CW complex of dimension $\le n-r$, so we may as well assume this is the case to begin with. If we assume moreover that $n \le 3r-1$ then the Freudenthal suspension theorem implies that L desuspends, so we may write $L \simeq \Sigma K$, and X is then of the form $\Sigma K \cup_\alpha D^n$ up to homotopy. Hence the corollary applies in this instance. Lastly, if we assume that $n \le 3r-2$, then K is unique up to homotopy.

The above result shows that it would be too optimistic to expect an algebraic classification of Poincaré complexes in the metastable range (indeed, the classification of self-dual CW complexes in the stable range would probably have to appear in any such classification). However, if we assume that we are at the very beginning of the metastable range, i.e., $n = 2r$, then ΣK is homotopy equivalent a wedge of r-spheres, say

$$\Sigma K \quad = \quad \bigvee^t S^r\,.$$

The Hilton decomposition [Hilt] can be used to write the homotopy class of α in terms of summands and basic Whitehead products, i.e,

$$\alpha \quad = \quad \sum_{j=1}^t \beta_j \iota_j \quad \oplus \quad \sum_{1\le i<j\le t} \gamma_{ij}[\iota_i, \iota_j]\,,$$

where $\iota_j\colon S^r \to \Sigma K$ denotes the (homotopy class of) the inclusion into the j-th summand, $\beta_j \in \pi_{n-1}(S^r)$ is an element, $[\iota_i, \iota_j] \in \pi_{n-1}(S^r \vee S^r)$ denotes the basic Whitehead product (= the attaching map $S^{2r-1} \to S^r \vee S^r$ for the top cell of $S^r \times S^r$) and γ_{ij} is an integer. Higher order Whitehead products do not appear in this formula for dimensional reasons.

It follows that the data (β_i, γ_{ij}) is a complete list of invariants for X. If e_j denotes the Kronecker dual to the cohomology class defined by ι_j, then the cohomology ring for X is given by

$$e_i \cup e_j \quad := \quad \begin{cases} \gamma_{ij} & \text{if } i < j, \\ (-1)^r \gamma_{ij} & \text{if } j < i, \\ H(\beta_i) \in \pi_{n-1}(S^{n-1}) = \mathbb{Z} & \text{if } i = j\,. \end{cases}$$

Therefore, the obstruction to X satisfying Poincaré duality is given by the demanding that matrix $(e_i \cup e_j)$ be invertible.

For the classification (of manifolds) in the odd dimensional case $n = 2r+1$, see [Wa3].

5. Poincaré embeddings

The notion of Poincaré embedding is a homotopy-theoretic impersonation of what one obtains from an embedding of actual manifolds. If a manifold X is decomposed as a union

$$X \quad = \quad K \cup_A C$$

where $K, C \subset X$ are codimension zero submanifolds with common boundary $A := K \cap C$, then X stratifies into two pieces, with A as the codimension one stratum and $\operatorname{int}(K \amalg C)$ as the codimension zero stratum. By replacing the above amalgamation with its homotopy invariant analogue, i.e., the homotopy colimit of $K \leftarrow A \to C$, we may recover X up to homotopy equivalence.

A Poincaré embedding amounts to essentially these data, except that we do not decree the spaces to be smooth manifolds: the manifold condition is weakened to the constraint that Poincaré duality is satisfied.

Specifically, suppose that we are given a connected based finite CW complex K^k of dimension k, a Poincaré n-complex X^n and a map $f\colon K \to X$. The definition of Poincaré embedding which we give is essentially due to Levitt [Le1].

5.1. Definition. We say that f *Poincaré embeds* if there exists a commutative diagram of based spaces

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle i}\big\downarrow & & \big\downarrow \\ K & \underset{f}{\longrightarrow} & X \end{array}$$

such that

- the diagram is a homotopy pushout, i.e., the evident map from the double mapping cylinder $K \times 0 \cup A \times [0,1] \cup C \times 1$ to X is a homotopy equivalence, and

- the image of $[X]$ under $H_n(X; {}^w\mathbb{Z}) \to H_n(i; f^{*w}\mathbb{Z})$ induced by the boundary map in Mayer-Vietoris sequence of the diagram gives $(\bar{K}, A)$ the structure of an n-dimensional Poincaré pair, where $\bar{K} := K \cup_{A\times 0} A\times[0,1]$ denotes the mapping cylinder of i. Similary, $[X]$ makes $(\bar{C}, A)$ into a Poincaré pair.
- The map i is $(n-k-1)$-connected.

The space C is called the *complement.*

The above definition applies when X has no boundary. If $(X, \partial X)$ is a Poincaré n-pair, then the definition is analogous, except that we require the map $\partial X \to X$ to factor as $\partial X \to C \to X$.

The first condition of the definition says that X is homotopy theoretically a union of K with its complement. The second condition says that the 'stratification' of X is 'Poincaré'. The last condition is essentially technical. In the smooth category, it would be an automatic consequence of transversality (a closed regular neighborhood N a k-dimensional subcomplex of an n-manifold has the property that $\partial N \subset N$ is $(n-k-1)$-connected), so the condition that i be $(n-k-1)$-connected is imposed to repair the lack of transversality in the Poincaré case. However, note when $k \le n-3$ that i is 2-connected if and only if i is $(n-k-1)$-connected, by duality and the relative Hurewicz theorem.

We will assume throughout that we are in codimension ≥ 3, i.e., $k \le n-3$.

5.2. Remark. Suppose additionally that K^k has the structure of a Poincaré k-complex. Then application of 3.4 above shows that the homotopy fibre of i is homotopy equivalent to an $(n-k-1)$-sphere. Hence the map i in the definition may be replaced by a spherical fibration. This recovers the notion of Poincaré embedding given by Wall [Wa2, p. 113].

The following result, which has a 'folk' co-authorship, says that the descent problem for finding locally flat PL-manifold embeddings can always be solved in codimension ≥ 3. Moreover, the smooth version can always be solved in the metastable range.

5.3. Theorem. *(Browder-Casson-Sullivan-Wall* [Wa2, 11.3.1]*). (1). Suppose that K^k and X^n are PL manifolds and that $k \le n-3$. Then f is homotopic to a locally flat PL embedding if and only if f Poincaré embeds.*

(2). If K^k, X^n are smooth manifolds with $k \le n-3$, then $f\colon K^k \to X^n$ is homotopic to a smooth embedding if and only if f Poincaré embeds and, additionally, one of the following holds: (i) $2n \ge 3(k+1)$, or (ii) f is homotopic to a smooth immersion.

Thus, the problem of finding an embedding of PL-manifolds in codimension ≥ 3 has been reduced to a problem in homotopy theory. *When can this homotopy problem be solved?*

A map $M^m \to N^n$ of manifolds with $n \ge 2m+1$ is always homotopic to an embedding, by transversality. It is natural to ask whether a similar result holds in the Poincaré case. Fix a map $f\colon K^k \to X^n$, where K^k is a k-dimensional

CW complex, X^n is a Poincaré complex (possibly with boundary) and $k \leq n{-}3$. According to Levitt [Le1], f Poincaré embeds when $n \geq 2k+2$. One would expect that the result holds in one codimension less, in analogy with manifolds, but this isn't known in general. However, Hodgson [Ho1] asserts that f will Poincaré embed when $n \geq 2k+1$, with the additional assumptions that K is a Poincaré complex and X is 1-connected. Both Hodgson and Levitt used manifold engulfing techniques to arrive at these results.

Recently, the author [Kl2] proved a general result about Poincaré embeddings which implies the Levitt and Hodgson theorems as special cases:

5.4. Theorem. *Let $f\colon K^k \to X^n$ be an r-connected map with $k \leq n{-}3$. Then f Poincaré embeds whenever*

$$r \quad \geq \quad 2k - n + 2\,.$$

Moreover, the Poincaré embedding is 'unique up to isotopy' if strict inequality holds.

(Two Poincaré embedding diagrams for f are called *isotopic* if they are isomorphic in the homotopy category of such diagrams.)

In contrast with the engulfing methods of Levitt and Hodgson, the author proves this result using purely homotopy theoretic techniques (a main ingredient of the proof is the Blakers-Massey theorem for cubical diagrams of spaces, as to be found in [Good]).

An old question about Poincaré complexes is whether or not the diagonal $X \to X \times X$ Poincaré embeds. As an application of the above, we have

5.5. Corollary. *Let X^n be a 2-connected Poincaré n-complex. Then the diagonal $X \to X \times X$ Poincaré embeds. Moreover, any two Poincaré embeddings of the diagonal are isotopic whenever X is 3-connected.*

It would be interesting to know whether or not the corollary holds without the connectivity hypothesis. Clearly, the diagonal of a manifold Poincaré embeds, by the tubular neighborhood theorem, so the existence of a diagonal Poincaré embedding for a Poincaré complex is an obstruction to finding a smoothing.

5.6. Example. Let X be a finite H-space with multiplication $\mu\colon X \times X \to X$. Write $X = X_0 \cup D^n$ using the disk theorem, and let $\alpha\colon D^n \to X$ be the characteristic map for the top cell of X. Consider the commutative diagram

$$\begin{array}{ccc}
X \times S^{n-1} & \xrightarrow{\mathrm{id}\times\alpha} & X \times X_0 \\
{\scriptstyle\cap}\big\downarrow & & \big\downarrow{\scriptstyle s} \\
X \times D^n & \xrightarrow{d} & X \times X
\end{array}$$

where the map s is given by $(x,y) \mapsto (x,\mu(x,y))$, and the map d is given by $(x,v) \mapsto (x, x\alpha(v))$. Then the diagram is a homotopy pushout and, moreover, the restriction of d to $X \times * \subset X \times D^n$ coincides with the diagonal. Hence, the diagram amounts to a Poincaré embedding of the diagonal.

5.7. Poincaré embeddings and unstable normal invariants. Another type of question which naturally arises concerns the relationship between the Spivak normal fibration and Poincaré embeddings in the sphere. Suppose that K^k is a Poincaré complex equipped with a choice of spherical fibration $p\colon S(p) \to K$ with fibre S^{j-1}. One can ask whether K^k Poincaré embeds in the sphere S^{k+j} with normal data p. That is, when does there exist a space W and an inclusion $S(p) \subset W$ such that $K \cup_{S(p)} W$ is homotopy equivalent to S^{k+j}? Obviously, if p isn't a Spivak fibration then there aren't any such Poincaré embeddings. So the first obstruction is given by the existence of a normal invariant $S^{k+j} \to T(p)$.

More generally, let K^k be a k-dimensional CW complex which is equipped with a map $g\colon A \to K$. Let $\bar{K}$ be the mapping cylinder of g and assume that $(\bar{K}, A)$ is an oriented Poincaré n-pair. We want to know when there exists a Poincaré embedding of K in S^n with normal data $A \to K$, i.e., when does there exist an inclusion of spaces $A \subset W$ such that $K \cup_A W$ is homotopy equivalent to S^n? This problem specializes to the previous one by taking g to be a spherical fibration.

Now, if the problem could be solved, then a choice of homotopy equivalence $S^n \xrightarrow{\simeq} K \cup_A W$ gives rise to a 'collapse' map

$$S^n \xrightarrow{\simeq} K \cup_A W \simeq \bar{K} \cup_A W \to \bar{K} \cup_A * = T(g)$$

where $T(g)$ denotes the mapping cone of $g\colon A \to K$. By correctly choosing our orientation for $(\bar{K}, A)$, we may assume that this map is of degree one. This prompts the following more general notion of normal invariant.

5.8. Definition. Given $g\colon A \to K$ as above together with an orientation for $(\bar{K}, A)$, we call the homotopy class of any degree one map $S^n \to T(g)$ a *normal invariant.*

The following result says that there is a bijective correspondence between normal invariants and isotopy classes of Poincaré embeddings in the sphere with given normal data in the metastable range. It was first proven by Williams [Wi1], using manifold methods. A homotopy theoretic proof has been recently given by Richter [Ri1].

5.9. Theorem. *Suppose that* $3(k+1) \le 2n$ *and* $n \ge 6$*. Then* K^k *Poincaré embeds in* S^n *with normal data* $g\colon A \to K$ *if and only if there exists a normal invariant* $S^n \to T(g)$*. Moreover, any two such Poincaré embeddings of* K *which induce the same normal invariants are isotopic provided that* $3(k+1) < 2n$.

Richter [Ri2] has found some interesting applications of this result. For example, he has shown how it implies that the isotopy class of a knot $S^n \subset S^{n+2}$ is determined by its complement X, whenever $\pi_*(X) = \pi_*(S^1)$ for $* \le 1/3(n+2)$; this extends a theorem of Farber by one dimension.

6. Poincaré Surgery

Controversy seems to be one of the highlights of this subject, so to avoid potential crossfire I'll begin this section with a quote from Chris Stark's mathematical review [Stk] of the book *Geometry on Poincaré spaces,* by Hausmann and Vogel [H-V]:

> *The considerable body of work on these matters is usually referred to as "Poincaré surgery" although other fundamental issues such as transversality are involved. These efforts involve several points of view and a number of mathematicians—the authors of the present notes identify three main streams of prior scholarship in their introduction and include a useful bibliography. Because of technical difficulties and unfinished research programs, Poincaré surgery has not become the useful tool proponents of the subject once hoped to deliver.*

For the sake of simplicity, I shall only discuss the results found in [H-V], which is now the standard reference for Poincaré surgery. We begin by explaining the fundamental problem of Poincaré surgery. To keep the exposition simple, we only consider the oriented case.

6.1. Surgery. Quinn [Qu2] defines a *normal space* to be a CW complex X equipped with an (oriented) $(k-1)$-spherical fibration $p_X\colon E \to X$ and a degree one map $\alpha_X\colon S^{n+k} \to T(p_X)$, where $T(p_X)$ denotes the mapping cone = Thom space of p_X (here the integer k is allowed to vary). We define the *formal dimension* of X to be n. Similarly, we have the notion of *normal pair* (X, A).

A *normal map* of normal spaces from X to Y consists of a map $f\colon X \to Y$ and an oriented fibre equivalence of fibrations $b\colon p_X \xrightarrow{\simeq} p_Y$ covering f such that the composite

$$S^{n+k} \xrightarrow{\alpha_X} T(p_X) \xrightarrow{T(b)} T(p_Y)$$

coincides with α_Y. Note that the mapping cylinder of f has the structure of a normal pair whose boundary is $X \amalg Y$. Similarly, there is an evident notion of *normal cobordism* for normal maps.

The obvious example of a normal space is given by a Poincaré complex equipped with Spivak fibration. The central problem of Poincaré surgery is to decide when a given normal map $f\colon X \to Y$ of Poincaré complexes is normally cobordant to a homotopy equivalence. Analogously, in the language of normal pairs, one wants to know when a normal pair (X, A), with A Poincaré, is normally cobordant to a Poincaré pair.

The algebraic theory of surgery of Ranicki [Ra1-2], [Ra3] associates to a normal map of Poincaré complexes $f\colon X \to Y$ a surgery obstruction $\sigma(f) \in L_n(\pi_1(Y))$ which coincides with the classical one if the given normal map comes from a manifold surgery problem. The principal result of Poincaré surgery says that this is the only obstruction to finding such a normal cobordism, i.e., that the manifold and Poincaré surgery obstructions are the same. According to Hausmann and Vogel, there are to date three basic approaches to Poincaré surgery obstruction theory.

The first is to use thickening theory to replace a Poincaré complex with manifold with boundary, so that we can avail ourselves of manifold techniques, such as engulfing. This is the embodied in approach of several authors, including Levitt [Le2], Hodgson [Ho6] and Lannes-Latour-Morlet [L-L-M]. One philosophical disadvantage of this approach is that, in the words of Browder, "a problem in homotopy theory should have a homotopy theoretical solution" [Qu1].

The second approach, undertaken by Jones [Jo1], also uses sophisticated manifold theory. The idea here is to equip Poincaré complexes with the structure of a *patch space,* which a space having an 'atlas' of manifolds whose transition maps are homotopy equivalences, and having suitable transversality properties.

Lastly, we have the direct homotopy theoretic assault, which was first outlined by Browder and which was undertaken by Quinn [Qu1-3]. If a map $\beta\colon S^j \to X^n$ is an element on which one wants to do surgery, then the homotopy cofiber $X \cup_\beta D^{j+1}$ has the homotopy type of an elementary cobordism, i.e., the trace of the would-be surgery. Moreover, as Quinn observes, if the surgery can be done then there is a cofibration sequence $X' \to X \cup_\beta D^{j+1} \to S^{n-j}$ where X' is the 'other end' of the cobordism. The composite map $X \subset X \cup_\beta D^{j+1} \to S^{n-j}$ is a geometric representative for a cohomology class which is Poincaré dual to the homology class defined by β. Quinn's idea [Qu3] is to find homotopy theoretic criteria (involving Poincaré duality) to decide when a map $X \cup_\beta D^{j+1} \to S^{n-j}$ extends to the left as a cofibration sequence, thus yielding X'.

Hausmann and Vogel point out that these three approaches are imbued with a great deal of technical difficulty and none of them were completely overcome. We pigeonhole the book of Hausmann and Vogel by placing it within the first of these schools.

6.2. Poincaré bordism. Under this title belong the fundamental exact sequences of Poincaré bordism found by Levitt [Le2], Jones [Jo1] and Quinn [Qu2]. Given a normal space X, we can let $\Omega_n^P(X)$ denote the bordism group of normal maps $(f,b)\colon Y \to X$ with X a normal space of formal dimension n and Y a Poincaré n-complex, and where cobordisms are understood in the Poincaré sense. Similarly, we can define $\Omega_n^N(X)$ to be the bordism group of normal maps $(f,b)\colon Y \to X$. Then there is an exact sequence

$$\cdots \to L_n(\pi_1(X)) \to \Omega_n^P(X) \xrightarrow{\text{incl}} \Omega_n^N(X) \to L_{n-1}(\pi_1(X)) \to \cdots$$

and moreover, an isomorphism $\Omega_n^N(X) \cong H_n(X; MSG)$, where the latter denotes the homology of X with coefficients in the Thom spectrum MSG whose n-th space is the Thom space of the oriented spherical fibration with fibre S^{n-1} over the classifying space BSG_n.

6.3. Transversality. Let A be a finite CW complex and suppose that (D,S) is a connected CW pair such that A includes in D as a deformation retract. We also assume that the homotopy fibre of $S \subset D$ is $(k-1)$-spherical. Given an inclusion $S \subset C$, let Y denote the union $D \cup_S C$. Roughly, we are thinking of the Y as containing a 'neighborhood thickening' D of A in such a way that the 'link' S of A in Y is a spherical fibration (up to homotopy).

Let X be a Poincaré n-complex and let $f\colon X \to Y$ be a map. We say that f is *Poincaré transverse* to A when $(f^{-1}(D), f^{-1}(S))$ and $(f^{-1}(C), f^{-1}(S))$ have the structure of Poincaré n-pairs, and moreover, we require that the homotopy fibre of the map

$$f^{-1}(S) \to f^{-1}(D)$$

is also $(k-1)$-spherical.

Hence, if f is Poincaré transverse to A, we obtain a stratification of X as a union of $f^{-1}(D)$ with $f^{-1}(C)$ along a common Poincaré boundary $f^{-1}(S)$. Moreover, it follows from the definition that $f^{-1}(A)$ has the structure of a Poincaré $(n-k)$-complex, so we infer that the inclusion $f^{-1}(A) \subset X$ Poincaré embeds (with normal data $f^{-1}(S)$).

The main issue now is to decide when a map $f\colon X \to Y$ can be 'deformed' (bordant, h-cobordant) so that it becomes Poincaré transverse to the given A. The philosophy is that although one can always deform a map in the smooth case to make it transverse, there are obstructions in the Wall L-groups for the Poincaré case, and the vanishing of these obstructions are both necessary and sufficient for Poincaré transversality up to bordism.

The algebraic L-theory codimension k Poincaré transversality obstructions for $k = 1, 2$ are discussed in Ranicki [Ra3, Chap. 7]. Supposing in what follows that $k \geq 3$, Hausmann and Vogel provide a criterion for deciding when f can be made (oriented) Poincaré bordant to a map which is transverse to A [H-V, 7.11]. They define an invariant $t(f) \in L_{n-k}(\pi_1(A))$ whose vanishing is necessary and sufficient to finding the desired bordism. If in addition f is 2-connected, then $t(f)$ is the complete obstruction to making f transverse to A up to homotopy equivalence (i.e., *Poincaré h-cobordism*) [loc. cit., 7.23]). Assertions of this kind can be found in the papers of Levitt [Le2],[Le4],[Le5], Jones [Jo1], and Quinn [Qu2]. For a general formulation, see [H-V, 7.11, 7.14].

6.4. Handle decompositions. Given a Poincaré n-pair $(Y, \partial Y)$, and a Poincaré embedding diagram

$$\begin{array}{ccc} S^{k-1} \times S^{n-k-1} & \longrightarrow & C \\ \cap\big\downarrow & & \big\downarrow \\ S^{k-1} \times D^{n-k} & \longrightarrow & \partial Y \end{array}$$

we can form the Poincaré n-pair

$$(Z, \partial Z) \quad := \quad (Y \cup D^k \times D^{n-k}, C \cup D^k \times S^{n-k-1}),$$

where $D^k \times D^{n-k}$ is attached to Y by means of the composite $S^{k-1} \times D^{n-k} \to \partial Y \subset Y$ and $D^k \times S^{n-k-1}$ is attached to ∂Y by means of the map $S^{k-1} \times S^{n-k-1} \to C$. Call this operation the effect of *attaching a k-handle* to $(Y, \partial Y)$. Note that there is an evident map $Y \to Z$.

A *handle decomposition* for a Poincaré complex X^n consists of a sequence of spaces

$$W_{-1} \to W_0 \to \cdots \to W_n$$

(with $W_{-1} = \emptyset$) and a homotopy equivalence $W_n \xrightarrow{\simeq} X$. Moreover, each W_j is the underlying space of a Poincaré n-pair with boundary ∂W_j in such a way that W_j is obtained from W_{j-1} a a finite number of j-handle attachments. Handle decompositions are special cases of Jones' patch spaces [Jo1].

6.5. Theorem. *(*[H-V, 6.1]*). If X is a Poincaré n-complex with $n \geq 5$, then X admits a handle decomposition.*

6.6. Appendix: a quick update on the finite H-space problem

When Browder posed his question: *Does every finite H-space have the homotopy type of a closed smooth manifold?*, it wasn't known that there exist 1-connected finite H-spaces which are not the homotopy type of compact Lie groups (except for products with S^7 or quotients thereof; see Hilton-Roitberg [H-R] and Stasheff [Sta, p. 22] for examples).

We remarked in §3 that every 1-connected finite H-space X^n has the homotopy type of a closed topological n-manifold. Browder [Br5] has noted in fact that the manifold can be chosen as smooth and stably parallelizable if n isn't of the form $4k{+}2$.

Using Zabrodsky mixing [Z] and surgery methods, Pedersen [Pe] was able to extend Browder's theorem to show that certain classes of finite H-spaces (some with non-trivial fundamental group) have the homotopy type of *smooth* manifolds. Recall that spaces Y and Z are said to have the same *genus* if $Y_{(p)} \simeq Z_{(p)}$ for all primes p, where $Y_{(p)}$ denotes the Sullivan localization of Y at p. Among other things, Pedersen proved that when a finite H-space X happens to be 1-connected and has the genus of a 1-connected Lie group, then X has the homotopy type of a smooth, parallelizable manifold.

Weinberger [We] has settled the 'local' version of the problem: if P denotes a finite set of primes, then a finite H-space is P-locally homotopy equivalent to a closed topological manifold.

Using localization techniques and surgery theory, Cappell and Weinberger [CW1] have shown that a finite H-space X has the homotopy type of a closed topological manifold when $\pi_1(X)$ is either an odd p-group or infinite with at most cyclic 2-torsion. In another paper [CW2] they show that X has the homotopy type of a closed smooth parallelizable manifold whenever $X_{(2)}$ contains a factor which is S^7 or a Lie group, and moreover, $\pi_1(X)$ is either trivial, an odd p-group or infinite with no 2-torsion. It is perhaps worth remarking that all known examples of finite H-spaces are known to be of this kind.

References

[At] M. Atiyah, *Thom complexes*, Proc. London. Math. Soc. **11** (1961), 291–310.

[Bar] D. Barden, *Simply connected five-manifolds*, Ann. Math. **82** (1965), 365–385.

[Bauer] S. Bauer, *The homotopy type of a four-manifold with finite fundamental group*, Proc. Top. Conf. Göttingen 1987, Springer LNM 1361, 1988, pp. 1–6.

[Baues] H. J. Baues, *Combinatorial Homotopy and 4-dimensional Complexes*, Expositions in Mathematics, De Gruyter, 1991.

[B-S] J. M. Boardman and B. Steer, *On Hopf invariants*, Comment. Math. Helv. **42** (1967), 180–221.

[Br1] W. Browder, *Torsion in H-spaces*, Ann. Math. **74** (1961), 24–51.

[Br2] ———, *Embedding* 1*-connected manifolds*, Bull. AMS **72** (1966), 225–231, erratum 736.

[Br3] ———, *Embedding smooth manifolds*, Proc. ICM (Moscow 1966), Mir, 1968, pp. 712–719.

[Br4] ———, *Surgery on Simply-connected Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 65, Springer-Verlag, 1972.

[Br5] ———, *Homotopy type of differentiable manifolds (Arhus 1962)*, Novikov Conjectures, Index Theorems and Rigidity (Oberwolfach, 1993), Vol. 1, London Math. Soc. Lecture Note Ser., vol. 226, Cambridge Univ. Press, 1995, pp. 97–100.

[Br-Sp] W. Browder and E. Spanier, *H-spaces and duality*, Pacific J. Math. **12** (1962), 411–414.

[C-W1] S. Cappell and S. Weinberger, *Which H-spaces are manifolds? I.*, Topology **27** (1988), 377–386.

[C-W2] ———, *Parallelizability of finite H-spaces*, Comment. Math. Helv. **60** (1985), 628–629.

[Di] J. Dieudonné, *A History of Algebraic and Differential Topology*, Birkhäuser, 1988.

[E-L] B. Eckmann and P. Linnell, *Poincaré duality groups of dimension two. II*, Comment. Math. Helv. **58** (1983), 111–114.

[E-M] B. Eckmann and H. Müller, *Poincaré duality groups of dimension two*, Comment. Math. Helv. **55** (1980), 510–520.

[G-S] S. Gitler and J. Stasheff, *The first exotic class of BF*, Topology **4** (1965), 257–266.

[Good] T. Goodwillie, *Calculus II, analytic functors*, K-theory **5** (1992), 295–332.

[Got] D. Gottlieb, *Poincaré duality and fibrations*, Proc. AMS **76** (1979), 148–150.

[H-K] I. Hambleton and M. Kreck, *On the classification of topological* 4*-manifolds with finite fundamental group*, Math. Ann. **280** (1988), 85–104.

[He] H. Hendriks, *Obstruction theory in* 3*-dimensional topology: an extension theorem*, J. London Math. Soc. (2) **16** (1977), 160–164.

[Hill] J. A. Hillman, *The Algebraic Characterization of Geometric* 4*-Manifolds.*, London Mathematical Society Lecture Note Series, vol. 198, Cambridge University Press, 1994.

[Hilt] P. J. Hilton, *On the homotopy groups of the union of spheres*, Jour. London Math. Soc. **30** (1955), 154–172.

[H-R] P. J. Hilton and J. Roitberg, *On principal S^3-bundles over spheres*, Ann. Math. **90** (1969), 91–107.

[Ho1] J. P. E. Hodgson, *The Whitney technique for Poincaré complex embeddings*, Proc. AMS **35** (1972), 263–268.

[Ho2] ———, *General position in the Poincaré duality category*, Invent. Math. **24** (1974), 311–334.

[Ho3] ———, *Subcomplexes of Poincaré complexes*, Bull. AMS **80** (1974), 1146–1150.

[Ho4] ———, *General position in the Poincaré duality category*, Invent. Math. **24** (1974), 311–334.

[Ho5] ———, *Trivializing spherical fibrations and embeddings of Poincaré complexes.*, Math. Zeit. **158** (1978), 35–43.

[Ho6] ———, *Surgery on Poincaré complexes*, Trans. Amer. Math. Soc. **285** (1984), 685–701.

[Jo1] L. E. Jones, *Patch spaces: a geometric representation for Poincaré spaces*, Ann. Math. **97** (1973), 306–343.

[Jo2] ———, *Corrections for "Patch Spaces"*, Ann. Math. **102** (1975), 183–185.

[Ke] M. Kervaire, *A manifold which does not admit a differentiable structure*, Comm. Math. Helv. **34** (1960), 257–270.

[K-M] M. Kervaire and J. Milnor, *Groups of homotopy spheres I.*, Ann. Math. **77** (1963), 504–537.

[Kl1] J. R. Klein, *On two results about fibrations*, to appear in Manuscripta Math.

[Kl2] ———, *On embeddings of complexes in Poincaré complexes*, in preparation.

[Kl3] ———, *On the homotopy embeddability of complexes in euclidean space I. the weak thickening theorem*, Math. Zeit. **213** (1993), 145–161.

[L-L-M] J. Lannes, L. Latour and C. Morlet, *Géométrie des complexes de Poincaré et chirurgie*, IHES preprint (1971).

[Le1] N. Levitt, *On the structure of Poincaré duality spaces*, Topology **7** (1968), 369–388.

[Le2] ______, *Poincaré duality cobordism*, Ann. Math. **96** (1972), 211–244.

[Le3] ______, *Applications of Poincaré duality spaces to the topology of manifolds*, Topology **11** (1972), 205–221.

[Le4] ______, *A general position theorem in the Poincaré duality category*, Comm. Pure Appl. Math. **25** (1972), 163–170.

[Le5] ______, *On a Σ-spectrum related to G/Top*, Houston J. Math. **3** (1977), 481–493.

[M-M] I. Madsen and R. J. Milgram, *The Classifying Spaces for Surgery and Cobordism of Manifolds*, Ann. of Math. Studies, vol. 92, Princeton University Press, 1979.

[May] J. P. May, *Math. Review of "Extension of maps as fibrations and cofibrations", by Frank Quinn*, MR 52♯6706.

[Mi1] J. Milnor, *On manifolds homeomorphic to the* 7*-sphere*, Ann. Math. **64** (1956), 399–405.

[Mi2] ______, *On simply-connected* 4*-manifolds*, Symposium Internacional de Topologı́a Algebraica (Mexico 1966), 1958, pp. 122–128.

[M-K-R] K. Myung Ho, S. Kojima and F. Raymond, *Homotopy invariants of non-orientable* 4*-manifolds.*, Trans. AMS **333** (1992), 71–81.

[N] S. P. Novikov, *Homotopy equivalent smooth manifolds*, Translations AMS **48** (1965), 271–396.

[Pe] E. K. Pedersen, *Smoothings of H-spaces*, Algebraic and Geometric Topology (Stanford 1976), Proc. Symp. Pure. Math., vol. 32(2), AMS, 1978, pp. 215–216.

[Qu1] F. Quinn, *Surgery on Poincaré spaces*, Mimeographed Notes, NYU, May 1971.

[Qu2] ______, *Surgery on Poincaré and normal spaces*, Bull. AMS **78** (1972), 262–267.

[Qu3] ______, *Extensions of maps as fibrations and cofibrations*, Transactions AMS **211** (1975), 203–208.

[Ra1] A. A Ranicki, *Algebraic theory of surgery I. Foundations*, Proc. London. Math. Soc. **40** (1980), 87–192.

[Ra2] ______, *Algebraic theory of surgery II. Applications to topology*, Proc. London. Math. Soc. **40** (1980), 193–287.

[Ra3] ______, *Exact Sequences in the Algebraic Theory of Surgery*, Mathematical Notes, vol. 26, Princeton University Press, 1981.

[Ra4] ______, *Algebraic L-theory and Topological Manifolds*, Cambridge Tracts in Mathematics, vol. 102, Cambridge University Press, 1992.

[Ri1] W. Richter, *A homotopy-theoretic proof of Williams's metastable Poincaré embedding theorem*, to appear in Duke Math. Jour.

[Ri2] ______, *High-dimensional knots with $\pi_1 \cong \mathbb{Z}$ are determined by their complements in one more dimension than Farber's range*, Proc. AMS **120** (1994), 285–294.

[Sm] S. Smale, *On the structure of manifolds*, Amer. J. Math. **84** (1962), 387–399.

[Spa] E. Spanier, *Algebraic Topology*, McGraw Hill, 1966.

[Spi] M. Spivak, *Spaces satisfying Poincaré duality*, Topology **6** (1967), 767–102.

[Stk] C. Stark, *Math. Review of "Geometry on Poincaré Spaces", by Hausmann and Vogel*, MR 95a:57042.

[Sta] J. Stasheff, *H-spaces from the homotopy point of view*, Springer LNM 161, 1970.

[Sto] R. Stöcker, *The structure of* 5*-dimensional Poincaré duality spaces*, Comm. Math. Helv. **57** (1982), 81–510.

[Su] D. Sullivan, *Triangulating and smoothing homotopy equivalences and homeomorphisms, (Geometric Topology Seminar Notes, Princeton University, 1967)*, The Hauptvermutung Book, Kluwer, 1996, pp. 69–103.

[Te] P. Teichner, *Topological* 4*-manifolds with finite fundamental group*, Dissertation, Mainz, 1992.

[Tu] V. Turaev, *Three-dimensional Poincaré complexes: classification and splitting*, Soviet Math. Dokl. **23** (1981), 312–314.

[Vo] W. Vogell, *The involution in the algebraic K-theory of spaces*, Springer LNM 1126, Proceedings, Rutgers 1983 (1985), 277–317.

[Wa1] C. T. C. Wall, *Classification of* $(n-1)$*-connected* $2n$*-manifolds*, Ann. Math. **75** (1962), 163–189.

[Wa2] ______, *Classification problems in differential topology. VI. Classification of* $(s-1)$*-connected* $(2s+1)$*-manifolds*, Topology **6** (1967), 273–296.

[Wa3] ______, *Poincaré complexes: I*, Ann. Math. **86** (1970), 213–245.

[Wa4] ______, *Surgery on Compact Manifolds*, Academic Press, 1970.

[We] S. Weinberger, *Homologically trivial group actions. I. Simply connected manifolds*, Amer. J. Math. 108 (1986), 1005–1021.

[Wi1] B. Williams, *Applications of unstable normal invariants–Part I*, Comp. Math. **38** (1979), 55–66.

[Wi2] ______, *Hopf invariants, localization and embeddings of Poincaré complexes*, Pacific J. Math. **84** (1979), 217–224.

[Z] A. Zabrodsky, *Homotopy associativity and finite CW complexes*, Topology **9**, 121–128.