# POINCARÉ EMBEDDINGS OF SPHERES

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Abstract. Given a 1-connected Poincaré duality space $M$ of dimension $2p$, with $p > 2$, we give criteria for deciding when homotopy classes $S^p \to M$ are represented by framed Poincaré embedded $p$-spheres.

## 1. Introduction

In setting up surgery theory in the smooth category in the even dimensional case, it is desirable to be able to decide when a homotopy class $S^p \longrightarrow M^{2p}$ is represented by a framed embedded sphere. Here $M$ denotes a compact smooth manifold of dimension $2p$, possibly with boundary. By transversality, any homotopy class is representable by an immersion, and the immersion has a trivial normal bundle if and only if its Euler characteristic vanishes. In [Wa2, Chap. 5], Wall described a 'self-intersection' invariant $\mu(x)$ for immersions $x \colon S^p \longrightarrow M$ whose vanishing is necessary and sufficient for finding an embedding in the same regular homotopy class as $x$. If $M$ is simply connected, Wall's invariant takes values in the abelian group $Q_p$, where $Q_p = \mathbb{Z}$ if $p$ is even, and $\mathbb{Z}/2$ if $p$ is odd.

In this paper we will concentrate on the Poincaré duality space version of this problem. From now on, $M$ will denote a 1-connected Poincaré duality space of dimension $2p$, possibly with boundary. We seek criteria for deciding when a map $g \colon S^p \longrightarrow M$ underlies a framed Poincaré embedding. The first obstruction concerns the existence of a framing: $g^*\nu_M$ should be fiber homotopically trivial, where $\nu_M$ is the Spivak normal fibration of $M$. This leads to

**Definition.** Consider pairs $(g, \tau)$, where $g \colon S^p \longrightarrow M$ is a map and $\tau$ is a (stable) trivialization of $g^*\nu_M$. Two such pairs $(g_0, \tau_0)$ and $(g_1, \tau_1)$ will be declared equivalent if there exists a homotopy $G \colon S^p \times I \longrightarrow M$ from $g_0$ to $g_1$ and a trivialization of $G^*\nu_M$ extending the given trivializations $\tau_0$ and $\tau_1$. Let $\mathcal{F}(S^p, M)$ denote the associated set of equivalence classes.

In fact $\mathcal{F}(S^p, M)$ has the structure of an abelian group if $p \geq 2$. To see this, choose a basepoint in $M$ and let $\mathcal{F}_*(S^p, M)$ be given by pairs $(g, \tau)$ but where now $g \colon S^p \longrightarrow M$ is a based map, and the equivalence relation is defined similarly except that homotopies are required to fix the

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basepoint. Then we have a forgetful function $\mathcal{F}_*(S^p, M) \to \mathcal{F}(S^p, M)$ which is an isomorphism since $M$ is 1-connected. But $\mathcal{F}_*(S^p, M)$ admits the structure of a group; the group structure is induced by the addition of homotopy classes. This group is abelian because $p \geq 2$. The map $\mathcal{F}(S^p, M) \to H_p(M)$ given by $(g, \tau) \mapsto g_*([S^p])$ is a homomorphism. Moreover, any framed Poincaré embedding of $S^p$ in $M$ defines an element of $\mathcal{F}(S^p, M)$.

**Theorem A.** *($p > 2$). There exists a function $\mu\colon \mathcal{F}(S^p, M) \to Q_p$ satisfying*

- $\mu(x + y) = \mu(x) + \mu(y) + x \cdot y$
- $x \cdot x = \mu(x) + (-1)^p \mu(x)$, *and*
- $\mu(ax) = a^2 \mu(x)$ *for all* $a \in \mathbb{Z}$.

*(here $x \cdot y$ is the intersection of the images of $x$ and $y$ in $H_p(M)$; intersection is defined using cup product and Poincaré duality).*

*Moreover, a class $x \in \mathcal{F}_p(M)$ is represented by a framed Poincaré embedded sphere if and only if $\mu(x) = 0$.*

The proof of Theorem A relies on the embedding result "Theorem E" of [Kl1], the latter which in turn was proved using fiberwise homotopy theory. After [Kl1] was written, Bill Richter and I discovered a non-fiberwise proof of "Theorem E."

For applications to Poincaré surgery one needs the above result for elements $x$ in the homology kernel of a highly connected Poincaré normal map. More precisely, suppose that $X^{2p}$ is a 1-connected Poincaré duality space with Spivak normal fibration $\nu_X$. One says that a degree one map $f\colon (M, \partial M) \to (X, \partial X)$ is a *normal map* if the restriction to $\partial M \to \partial X$ is a homotopy equivalence, and $f$ comes equipped with a fiber homotopy equivalence $\nu_M \cong f^*\xi$, where $\nu_M$ is the stable normal bundle of $M$. There is then the corresponding notion of *normal cobordism*, and the fundamental problem of Poincaré surgery is to decide when $f$ can be made normally cobordant to a homotopy equivalence. By doing 'surgery below the middle dimension', it can be arranged that $f\colon M \to X$ is $p$-connected. The relative Hurewicz theorem gives an isomorphism $\pi_{p+1}(f) \cong H_{p+1}(f)$, and Poincaré duality implies that $f$ is a homotopy equivalence if and only if $H_{p+1}(f)$ is trivial, i.e., if and only if $f$ is $(p{+}1)$-connected. Thus one wants to know when $f$ can be made normally cobordant to a $(p{+}1)$-connected map.

Setting

$$K_p(M) := \pi_{p+1}(f) \cong H_{p+1}(f) = \mathrm{kernel}(H_p(M) \to H_p(X))\,.$$

we have a monomorphism $K_p(M) \to \mathcal{F}(S^p, M)$ defined by $x \mapsto (g_x, \tau_x)$ in which $x$ represents an element of $\pi_{p+1}(f)$ given as a map $g_x\colon S^p \to M$ and a null-homotopy $D^{p+1} \to X$ of $f \circ g_x$, and $\tau_x$ is the trivialization of $g_x^*\nu_M = (f \circ g_x)^*\nu_X$ defined by its extension to $D^{p+1}$. Then the function $\mu\colon K_p(M) \to Q_p$ gives the obstruction to representing elements of $K_p(M)$ by framed Poincaré embedded spheres. The triple $(K_p(M), \lambda, \mu)$ in which $\lambda\colon K_p(M)\times K_p(M) \to \mathbb{Z}$ is the intersection form define an element $\sigma(f)$ of the $L$-group $L_{2p}(e)$ and the fundamental theorem of Poincaré surgery says that the vanishing of $\sigma(f)$ is necessary and sufficient for finding a Poincaré normal cobordism to a homotopy equivalence.

Bill Richter and I plan to show in a future paper how Theorem A implies the fundamental theorem of Poincaré surgery in the 1-connected case. We will also show how to generalize Theorem A to the non-simply connected case.

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## 2. Preliminaries

**Spaces.** Our ground category is **Top**, the category of compactly generated Hausdorff spaces. This comes equipped with the structure of a Quillen model category:

- The *weak equivalences* are the weak homotopy equivalences (i.e., maps $X \to Y$ such that the associated realization of its singular map $|S.X| \to |S.Y|$ is a homotopy equivalence). Weak equivalences are denoted $\xrightarrow{\sim}$.
- The *fibrations*, denoted $\twoheadrightarrow$, are the Serre fibrations.
- The *cofibrations*, denoted $\rightarrowtail$, are the 'Serre cofibrations', i.e., inclusion maps given by a sequence of cell attachments (i.e., relative cellular inclusions) or retracts thereof.

Every object is fibrant. The cofibrant objects are the retracts of iterated cell attachments built up from the empty space. Every object $Y$ comes equipped with a functorial cofibrant approximation $Y^{\mathrm{c}} \xrightarrow{\sim}\!\!\!\!\twoheadrightarrow Y$.

A non-empty space is always $(-1)$-connected. A connected space is 0-connected, and is $r$-connected for some $r > 0$ if its homotopy groups vanish up through degree $r$, for any choice of basepoint. A map of non-empty spaces $X \to Y$ is called *$r$-connected* if its homotopy fiber with respect to any choice of basepoint in $Y$ is an $(r-1)$-connected space. An $\infty$-connected map is a weak equivalence.

A space is *homotopy finite* if it is homotopy equivalent to a finite CW complex.

We will making use of homotopy pushouts in **Top**, which are to be formed using double mapping cylinders. If $Y \leftarrow X \to Z$ is a diagram of cofibrant spaces, its homotopy pushout is $Y\times 0 \cup X\times[0,1] \cup Z\times 1$. If the spaces in the diagram fail to be cofibrant, to guarantee a homotopy invariant construction one applies cofibrant replacement to the spaces in the diagram before taking the double mapping cylinder. If $Y \to X$ is a map of cofibrant spaces, then the homotopy pushout of the diagram $X \leftarrow Y \to X$ is the *(unreduced) fiberwise suspension* $\Sigma_X Y$ of $Y \to X$; it comes equipped with an evident map $\Sigma_X Y \to X$.

If $Y \to X$ is a map of cofibrant spaces, we will often write $(\bar{X}, Y)$ for the cofibration pair $(X \cup_{Y\times 0} Y \times [0,1], Y\times 1)$ associated with the mapping cylinder.

A commutative diagram of spaces

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is *cocartesian* (née homotopy cocartesian) if the induced map from the homotopy pushout of $C \leftarrow A \to B$ to $D$ is a weak equivalence.

If $X$ and $Y$ are based spaces, which are cofibrant in the based sense (i.e., the inclusion of the basepoint is a cofibration of **Top**), then the *smash product* $X \wedge Y$ is given by collapsing the subspace $X \times * \cup * \times Y$ of $X \times Y$ to a point. In particular, we have interated smash products $X^{[n]} := X \wedge X^{[n-1]}$, where $X^{[1]} := X$.

Finally, a note about usage. In the majority of instances below, the term 'space' will refer to a cofibrant object of **Top**. However, there is one notable exception: in dealing with spherical fibrations $\xi \colon S(\xi) \to X$, we will *not* require the total space $S(\xi)$ to be cofibrant. The reason for this is that we will be forming base changes $S(\xi)_{|Y} \to Y$ along maps $Y \to X$ (here $S(\xi)_{|Y} = Y \times_X S(\xi)$ is one of the notations we will be using for the fiber product); cofibrancy of total spaces is not usually invariant under base changes.

**Thom spaces.** Given a spherical fibration $\xi \colon S(\xi) \to X$ (with $X$ cofibrant), the *Thom space* $X^\xi$ is the mapping cone of the composite $S(\xi)^{\mathrm{c}} \xrightarrow{\sim} S(\xi) \to X$. It is cofibrant when considered as a based space.

A map of spaces $f \colon Y \to X$ induces a based map of Thom spaces $Y^{f^*\xi} \to X^\xi$. where $f^*\xi$ denotes the base change of $\xi$ along $f$. More generally, given another spherical fibration $\zeta$ over $Y$ and a fiber homotopy equivalence $\zeta \xrightarrow{\simeq} f^*\xi$, we obtain a based map of Thom spaces $Y^\zeta \to X^\xi$ which we call the *thomification* of $f$.

**Poincaré spaces.** An *(oriented) Poincaré space* $X$ of dimension $n$ is a pair $(X, \partial X)$ such that $X$ and $\partial X$ are homotopy finite spaces, $\partial X \to X$ is a cofibration, and $X$ satisfies *Poincaré duality:*

- $X$ comes equipped with a fundamental class $[X] \in H_n(X, \partial X; \mathbb{Z})$ such that the cap product homomorphisms

$$\cap [X] \colon H^*(X) \to H_{n-*}(X, \partial X)$$

and

$$\cap[\partial X]\colon H^*(\partial X) \to H_{n-*-1}(\partial X)$$

are isomorphisms, where $[\partial X] \in H_{n-1}(\partial X)$ is the image of $[X]$ under the connecting homomorphism in the homology exact sequence of the pair $(X, \partial X)$; here coefficients are to be taken with respect to any coefficient bundle on $X$ (respectively on $\partial X$).

If both $X$ and $\partial X$ are simply connected, it is sufficient to check that Poincaré duality holds with integral coefficients. For the most part, we will be working with the simply connected case.

A Poincaré space $X$ is said to have *homotopy codimension* $\geq q$ (written $\operatorname{codim} X \geq q$) if

- The map $\partial X \to X$ is $(q-1)$-connected, and
- $X$ is homotopy equivalent to a CW complex of dimension $\leq n-q$.

A *Spivak fibration* $\nu\colon S(\nu) \to X$ is an oriented spherical fibration over $X$ which comes equipped with a based map $\alpha\colon S^{n+j} \to X^\nu/(\partial X)^{\nu_{|\partial X}}$ (where, say, the fiber of $\nu$ is $(j-1)$-spherical) satisfying the condition

$$U \cap \alpha_*([S^{n+j}]) = [X]\,.$$

Here $U \in H^j(X^\nu)$ is the Thom class. Since the stable fiber homotopy type of $\nu$ is unique [Wa1, 3.4], we will abuse language and refer to any choice of the above as *the* Spivak fibration.

## 3. Embeddings

Let $P^n$ and $M^n$ a Poincare spaces of dimension $n$, where $X$ is connected. An *embedding* of $P$ in $M$ is a commutative cocartesian square of homotopy finite spaces

$$\begin{array}{ccc} \partial P & \longrightarrow & C \\ {\scriptstyle \text{incl.}}\downarrow & & \downarrow {\scriptstyle g} \\ P & \underset{e}{\longrightarrow} & M \end{array}$$

together with a factorization of the inclusion $\partial M \to C \to M$, such that the composite

$$H_n(M, \partial M) \to H_n(\bar{M}, C) \cong H_n(P, \partial P)$$

is of degree one and the image of $[X]$ under the composite

$$H_n(M, \partial M) \to H_n(\bar{M}, P \amalg \partial M) \cong H_n(\bar{C}, \partial P \amalg \partial M)$$

equips $(\bar{C}, \partial P \amalg \partial M)$ with the structure of a Poincaré space (the second condition is automatic if $\operatorname{codim} P \geq 3$).

The space $C$ is called the *complement*, and $e\colon P \to M$ is the *underlying map* of the embedding. In this case, we say that the map $e\colon P \to M$ *embeds*. Sometimes we will have occasion to refer to the embedding by means of its underlying map.

Two embeddings from $P$ to $M$ with complements $C_0$ and $C_1$ are *elementary concordant* if there exists a commutative diagram of pairs

$$\begin{array}{ccc} ((\partial P)\times I, \partial P_0 \amalg \partial P_1) & \longrightarrow & (W, C_0 \cup (\partial M)\times I \cup C_1) \\ \Big\downarrow & & \Big\downarrow \\ (P\times I, P_0 \amalg P_1) & \longrightarrow & (M\times I, \partial(M\times I)) \end{array}$$

in which each associated diagram of spaces

$$\begin{array}{ccc} (\partial P)\times I & \longrightarrow & W \\ \Big\downarrow & & \Big\downarrow \\ P\times I & \longrightarrow & M\times I \end{array} \quad \text{and} \quad \begin{array}{ccc} \partial P_0 \amalg \partial P_1 & \longrightarrow & C_0 \cup (\partial M)\times I \cup C_1 \\ \Big\downarrow & & \Big\downarrow \\ P_0 \amalg P_1 & \longrightarrow & \partial(M\times I) \end{array}$$

is cocartesian, where the latter of these is obtained from the embedding diagrams using the inclusion $\partial M_0 \amalg \partial M_1 \subset \partial(M\times I)$. Moreover, we require the maps $C_i \to W$ to be weak equivalences (this is automatic whenever $\operatorname{codim} P \geq 3$).

More generally, *concordance* is the equivalence relation generated by elementary concordance.

The *decompression* of an embedding $e\colon P \to M$ with complement $C$ is the embedding $P\times I \to M\times I$ defined by the diagram

$$\begin{array}{ccc} \partial(P\times I) & \longrightarrow & W \\ \Big\downarrow & & \Big\downarrow \\ P\times I & \xrightarrow[e\times \mathrm{id}_I]{} & M\times I \end{array}$$

where $W = M_0 \cup C\times I \cup M_1 := \Sigma_M C$ is fiberwise suspension, and the factorization $\partial(M\times I) \to W \to M\times I$ is evident.

**Immersions.** A map $f\colon P^n \to M^n$ is said to *immerse* if the map $f\times \mathrm{id}_{D^j}\colon P\times D^j \to M\times D^j$ embeds for some integer $j \geq 0$ (such a representative embedding will be called an *immersion* of $f$). *Concordance* of immersions of $f$ is the equivalence relation generated by decompression and concordance of embeddings.

The following, which justifies our usage, is the Poincaré analogue of Smale-Hirsch theory:

**Theorem 3.1** (Klein [Kl2]). *A map $f\colon P \to M$ immerses if and only if there exists a fiber homotopy equivalence $\nu_P \simeq f^*\nu_M$.*

*Remark* 3.2. An embedding $f\colon P \to M$ with complement $C$ defines a fiber homotopy equivalence $\nu_P \simeq f^*\nu_M$ as follows: set $\nu = \nu_M$. The spherical class $S^{n+k} \to M^\nu/(\partial M)^{\nu_{|\partial M}}$ gives rise to a spherical class $S^{n+k} \to P^{f^*\nu}/(\partial P)^{f^*\nu_{|\partial M}})$ using

$$M^\nu/(\partial M)^{\nu_{|\partial M}} \;\to\; M^\nu/C^{\nu_{|C}} \;\simeq\; P^{f^*\nu}/(\partial P)^{f^*\nu_{|\partial M}}\,.$$

The uniqueness of the Spivak fibration then gives a fiber homotopy equivalence $\nu_P \simeq f^*\nu_M$. This procedure extends to immersions $f\colon P \to M$ in the obvious way.

Application of the preceding to the case $P = S^p \times D^p$ gives

**Corollary 3.3.** *Every element of $\mathcal{F}(S^p, M)$ is represented by an immersion from $P$ to $M$.*

**Definition 3.4.** Let $\mathcal{I}(P, M)$ be the concordance classes of immersions from $P$ to $M$.

When $P = S^p \times D^p$, we have a map

$$\mathcal{I}(P, M) \to \mathcal{F}(S^p, M)$$

given by $x \mapsto [x]$, where $[x]$ consists of the homotopy class of the underlying map $P \to M$ together with the stable trivialization of $x^*\nu_M$ defined by the fiber homotopy equivalence $\nu_P \simeq x^*\nu_M$, and the triviality of $\nu_P$.

The map $\mathcal{I}(P, M) \to \mathcal{F}(S^p, M)$ is surjective by 3.1 (see also to the addendum to Theorem A in [Kl2]).

## 4. The Thom-Pontryagin construction

Fix an embedding $P \to M$, with diagram

$$\begin{array}{ccc} \partial P & \longrightarrow & C \\ \big\downarrow & & \big\downarrow \\ P & \longrightarrow & M\,. \end{array}$$

The associated concordance class defines a homotopy class $\alpha \in [M/\partial M, P/\partial P]$, by means of the Thom-Pontryagin construction: replacing $C$ with a suitable mapping cylinder, if necessary, may assume that the map $\partial M \to C$ is a cofibration. Then $\alpha$ is given by

$$M/\partial M \xleftarrow{\sim} (P \cup_{\partial P} C)/\partial M \to (P \cup_{\partial P} C)/C = P/\partial P\,.$$

Similarly, given an immersion $P \to M$, we obtain a *stable* homotopy class

$$\alpha \in \{M/\partial M, P/\partial P\}$$

by applying the Thom-Pontryagin construction to a choice of representative embedding of $P \times D^j$ in $M \times D^j$ This is an invariant of the concordance class of the immersion.

The Thom-Pontryagin construction yields a function

$$\mathcal{I}(P, M) \xrightarrow{\text{t-p}} \{M/\partial M, P/\partial P\}\,.$$

It can be shown that the source admits a group structure in such a way that this map becomes a homomorphism, but we will not require this.

**The umkehr isomorphism.** Let $A$ and $B$ are based spaces which are homotopy finite. Assume $A$ and $B$ come equipped with Spanier-Whitehead $m$-duality maps $d_A \colon S^m \to A \wedge A^*$ and $d_B \colon S^m \to B \wedge B^*$. A well-known construction in homotopy theory gives an isomorphism of abelian groups

$$\{A, B\} \to \{B^*, A^*\}$$

called the *umkehr correspondence*, which is defined by the pair of isomorphisms

$$\{A, B\} \xrightarrow[\cong]{(-\wedge \mathrm{id}_{A^*})\circ d_A} \{S^m, B \wedge A^*\} \xleftarrow[\cong]{(\mathrm{id}_B \wedge -)\circ d_B} \{B^*, A^*\}\,.$$

For Poincaré spaces $P$ and $M$, apply the forgoing to the spaces $A = P^{\nu_P}$ and $B = M^{\nu_M}$. Since $A^* = P/\partial P$, $B^* = M/\partial M$, the umkehr correspondance gives an isomorphism

$$\{P^{\nu_P}, M^{\nu_M}\} \xrightarrow{\cong} \{M/\partial M, P/\partial P\}\,.$$

Consider now a pair $(g, \tau)$ representing an element of $\mathcal{F}(S^p, M)$. Setting $P = S^p \times D^p$, we obtain an identification of Thom spaces $(S^p)^{g^*\nu_M} \simeq P^{\nu_M}$. Consequently, the map $g \colon S^p \to M$ induces a map of Thom spaces $P^{\nu_P} \to M^{\nu_M}$, and we have a homomorphism

$$\mathcal{F}(S^p, M) \to \{P^{\nu_P}, M^{\nu_M}\}\,.$$

Composing the latter with the umkehr correspondance, we obtain a homomorphism

$$\mathcal{F}(S^p, M) \xrightarrow{\phi} \{M/\partial M, P/\partial P\}\,.$$

We have the following geometric interpretation of $\phi$:

**Proposition 4.1.** *The diagram*

$$\begin{array}{ccc} \mathcal{I}(P, M) & \xrightarrow{\text{t-p}} & \{M/\partial M, P/\partial P\} \\ \big| & & \\ & \phi & \\ \mathcal{F}(S^p, M) & & \end{array}$$

*is commutative, where* t-p *denotes the Thom-Pontryagin construction.*

The proof of the proposition will use the following

*Construction* 4.2. Let $K$ be a space, $\xi\colon S(\xi) \to K$ a spherical fibration and $E \to K$ a map of spaces. Set

$$\Sigma^{\xi}_K E \quad := \quad \operatorname{hocolim}\,(S(\xi) \leftarrow S(\xi)_{|E} \to E)$$

(this is a kind of twisted fiberwise suspension of $E \to K$). There is an evident map $\Sigma^{\xi}_K E \to K$. Furthermore, there is a cocartesian square

$$\begin{array}{ccc} \Sigma^{\xi}_K E & \longrightarrow & E^{\xi_{|E}} \\ \big\downarrow & & \big\downarrow \\ K & \longrightarrow & K^{\xi} \end{array}$$

which shows that the mapping cone of $\Sigma^{\xi}_K E \to K$ coincides up to homotopy with the mapping cone of $E^{\xi_{|E}} \to K^{\xi}$.

If $\epsilon_j\colon K \times S^{j-1} \to K$ denotes the trivial $(j-1)$-spherical fibration over $K$, then $\Sigma^{\epsilon_j}_K E$ is identified with the iterated fiberwise suspension $\Sigma^j_K E$.

*Proof of 4.1.* Given an element $x \in \mathcal{I}(P, M)$, we suppose first that $x$ is represented by an embedding $e\colon P \to M$ with complement $C$. Let $\alpha\colon M/\partial M \to P/\partial P$ denote the Thom-Pontryagin construction of $e$, let $e^{\nu}\colon P^{e^*\nu} \to M^{\nu}$ be the map of Thom spaces induced by $e$, where $\nu := \nu_M$.

Let $\partial D(\nu) := \Sigma^{\nu}_M \partial M$, and let $D(\nu)$ denote the mapping cylinder of the map $\partial D(\nu) \to M$. Then $D(\nu)$ is an $(n+j)$-dimensional Poincaré space with boundary $\partial D(\nu)$ (here $j$ denotes the integer such that the $\nu$ is $(j-1)$-spherical). Moreover, we have a degree one map

$$u\colon S^{n+j} \to D(\nu)/\partial D(\nu)\,.$$

The 'diagonal' $(D(\nu), \partial D(\nu)) \to (D(\nu) \times M, S(\nu) \times M \cup D(\nu) \times \partial M)$ induces a map

$$\Delta_{\nu}\colon D(\nu)/\partial D(\nu) \to M^{\nu} \wedge M/\partial M$$

upon passing to quotients. The composite $\Delta_{\nu} \circ u\colon S^{n+j} \to M^{\nu} \wedge M/\partial M$ is an $S$-duality map.

Similarly, we have an $(n+j)$-dimensional Poincaré space $D(e^*\nu)$ with boundary $\Sigma^{e^*\nu}_P \partial P$. Moreover, the evident map $D(e^*\nu) \to D(\nu)$ is the underlying map of an embedding, whose complement is $\Sigma^{\nu}_M C$. The Thom-Pontryagin construction of this embedding gives a map

$$\beta\colon D(\nu)/\partial D(\nu) \to D(e^*\nu)/\partial D(e^*\nu)\,.$$

Composing $\beta$ with $u$, we obtain a degree one map

$$u'\colon S^{n+j} \to D(e^*\nu)/\partial D(e^*\nu)\,.$$

Composing the diagonal map $\Delta_{e^*\nu}\colon D(e^*\nu)/\partial D(e^*\nu) \to P^{e^*\nu} \wedge P/\partial P$ with $u'$ yields an $S$-duality map $S^{n+j} \to P^{e^*\nu} \wedge P/\partial P$.

By definition of the umkehr isomorphism, what we need to show is that the $S$-dual of the map $e^\nu$ coincides with the $S$-dual of the map $\alpha$. More precisely, we need to show that the composite

$$S^{n+j} \xrightarrow{\Delta_{e^*\nu}\circ u'} P^{e^*\nu} \wedge P/\partial P \xrightarrow{e^\nu \wedge \mathrm{id}_{P/\partial P}} M^\nu \wedge P/\partial P$$

coincides up to homotopy with the composite

$$S^{n+j} \xrightarrow{\Delta_\nu \circ u} M^\nu \wedge M/\partial M \xrightarrow{\mathrm{id}_{M^\nu} \wedge \alpha} M^\nu \wedge P/\partial P\,.$$

Unraveling the definitions of these maps, it will be sufficient to show that the diagram

$$\begin{array}{ccc} D(\nu)/\partial D(\nu) & \xrightarrow{\quad\quad\Delta_\nu\quad\quad} & M^\nu \wedge M/\partial M \\ \Big\downarrow \beta & & \Big\downarrow \mathrm{id}_{M^\nu}\wedge\alpha \\ D(e^*\nu)/\partial D(e^*\nu) \xrightarrow[\Delta_{e^*\nu}]{} & P^{e^*\nu} \wedge P/\partial P \xrightarrow[e^\nu\wedge \mathrm{id}_{P/\partial P}]{} & M^\nu \wedge P/\partial P \end{array}$$

homotopy commutes (since prefixing the diagram with $u$ and then traversing the result in the two different possible ways gives the maps in question).

To see this, consider the commutative diagram of pairs

$$\begin{array}{ccc} (D(\nu), \partial D(\nu)) & \longrightarrow & (D(\nu)\times M, S(\nu)\times M \cup D(\nu)\times \partial M) \\ \Big\downarrow & & \Big\downarrow \\ (D(\nu), \Sigma^\nu_M C) & \longrightarrow & (D(\nu)\times M, S(\nu)\times M \cup D(\nu)\times C) \\ \Big\downarrow \sim & & \Big\downarrow \sim \\ (D(e^*\nu), \partial D(e^*\nu)) & \longrightarrow & (D(\nu)\times P, S(\nu)\times P \cup D(\nu)\times \partial P) \end{array}$$

where the horizontal arrows are given by diagonals, and the vertical ones are evident. The arrows labeled with '$\sim$' become weak equivalences of based spaces upon passage to quotients (more precisely, mapping cones). Upon passage to quotients, the left vertical line becomes the map $\beta$, and the right one becomes $\mathrm{id}_{M^\nu} \wedge \alpha$. The top arrow becomes $\Delta_\nu$ and the bottom one becomes $(e^\nu \wedge \mathrm{id}_{P/\partial P}) \circ \Delta_{e^*\nu}$. Since the diagram of pairs commutes, the corresponding diagram of quotients also commutes. But the diagram of quotients amounts to the next to last diagram above. This completes the argument in the case when $x \in \mathcal{F}(S^p, M)$ is represented by an embedding.

For the case of a general $x \in \mathcal{F}(S^p, M)$, we use a representative embedding $P\times D^\ell \to M\times D^\ell$ and argue as above, with $P$ replaced by $P\times D^\ell$ and $M$ replaced by $M\times D^\ell$. As this is straightforward, we omit the details. □

## 5. '$\mu = 0 \Rightarrow$ EMBEDDING'

Let $A$ and $B$ be based spaces. We write $D_2(B) := S^\infty_+ \wedge_{\mathbb{Z}/2} B^{[2]}$ for the quadratic construction of $B$. The *(stable) Hopf invariant* is a certain homotopy operation, natural in $A$ and $B$,

$$H_2 \colon \{A, B\} \to \{A, D_2(B)\} .$$

We recall its most important property:

**Theorem 5.1** (Milgram [Mi]). *Assume that $B$ is $r$-connected and that $A$ has the homotopy type of a CW complex of dimension $\leq 3r+1$. Then $\alpha \in \{A, B\}$ desuspends to an element of $[A, B]$ if and only if $H_2(\alpha) = 0$.*

**The invariant $\mu$.** We consider next the composite

$$\mathcal{F}(S^p, M) \xrightarrow{\phi} \{M/\partial M, P/\partial P\} \xrightarrow{H_2} \{M/\partial M, D_2(P)\} .$$

By an elementary calculation which we omit, $H_{2p}(D_2(P/\partial P))$ is canonically isomorphic to the abelian group

$$Q_p \quad := \quad \mathbb{Z}/(1 - (-1)^p)\mathbb{Z}$$

(so $Q_p = \mathbb{Z}$ when $p$ is even and $\mathbb{Z}/2$ when $p$ is odd).

Define a function

$$\mu \colon \mathcal{F}(S^p, M) \to Q_p$$

by

$$\mu(y) := (H_2 \circ \phi(y))_*([M]) .$$

The following is the main result concerning the existence of Poincaré framed embedded spheres representing elements of $\mathcal{F}(S^p, M)$:

**Theorem 5.2.** *Let $p > 2$. An immersion $x \colon P \to M$ compresses to an embedding if and only if $\mu([x]) = 0$. In particular, the elements of $\mathcal{F}(S^p, M)$ which are represented by embeddings of $P$ in $M$ are precisely those elements which map to zero under $\mu$.*

The proof is based on

**Theorem 5.3** (Klein [Kl1, Th. E], Richter). *An immersion $x \colon P \to M$, with $M$ 1-connected compresses to an embedding if and only if its stable collapse $\phi([x]) = \alpha \in \{M/\partial M, P/\partial P\}$ desuspends to an element of $[M/\partial M, P/\partial P]$.*

*Proof of 5.2.* By 5.1 and 5.3, $x \colon P \to M$ compresses to an embedding if and only if $H_2(\alpha) = 0$. For dimensional reasons and obstruction theory, the vanishing of $H_2(\alpha)$ is tantamount to the vanishing of $\mu([x])$ (since evaluating on the fundamental class defines an isomorphism $\{M/\partial M, D_2(P/\partial P)\} \cong H_n(D_2(P/\partial P))$). This establishes the first part.

The second part is a direct consequence of the first part together with the fact that every element of $\mathcal{F}(S^p, M)$ is represented by an immersion of $P$ in $M$ by 3.1. □

The next theorem summarizes the algebraic properties of $\mu$ (compare with the smooth case [Wa2, Th. 5.2]).

**Theorem 5.4.** *The function* $\mu\colon \mathcal{F}(S^p, M) \to Q_p$ *satisfies*

- $\mu(x+y) = \mu(x) + \mu(y) + x \cdot y$,
- $x \cdot x = \mu(x) + (-1)^p \mu(x)$, *and*
- $\mu(ax) = a^2 \mu(x)$ *for all* $a \in \mathbb{Z}$.

*Remark* 5.5. The intersection number $x \cdot y$ (defined via cup product and duality by considering $x$ and $y$ as homology classes) is to be interpreted in $Q_p$ by means of the reduction homomorphism $\mathbb{Z} \to Q_p$.

The third identity says that $\mu$ is a quadratic form.

*Proof of 5.4.* We will use the *unstable* Hopf invariant

$$h_2\colon [\Sigma A, \Sigma B] \to [\Sigma^2 A, (\Sigma B)^{[2]}]$$

of Boardman and Steer [B-S], a homotopy operation natural in $A$ and $B$. We recall only the facts about $h_2$ which are necessary for the proof:

- (Compatibility with $H_2$). The diagram

$$\begin{array}{ccc} [\Sigma A, \Sigma B] & \xrightarrow{h_2} & [\Sigma^2 A, (\Sigma B)^{[2]}] \\ \downarrow & & \downarrow \\ \{A, B\} & \xrightarrow[H_2]{} & \{A, D_2(B)\} \end{array}$$

  commutes (see Crabb [Cr, p. 61]), where the left vertical map is given by stablization and the right vertical map is given by the natural map $(\Sigma B)^{[2]} \to \Sigma^2 D_2(B)$ and stablization.
- (Cartan formula). For $\alpha, \beta \in [\Sigma A, \Sigma B]$ we have

$$h_2(\alpha + \beta) \quad = \quad h_2(\alpha) + \alpha \cdot \beta + h_2(\beta),$$

  where $\alpha \cdot \beta$ is the *cup product* of $\alpha$ and $\beta$, defined by the composite

$$\Sigma^2 A \xrightarrow{\Sigma^2 \Delta} \Sigma A \wedge \Sigma A \xrightarrow{\alpha \wedge \beta} \Sigma B \wedge \Sigma B$$

  where $\Delta\colon A \to A^{[2]}$ denotes the reduced diagonal map (see Boardman and Steer [B-S, 2.1]).
- (Transfer formula). If $\alpha \in [\Sigma A, \Sigma B]$ and $B$ is a suspension, then

$$\alpha \cdot \alpha \quad = \quad h_2(\alpha) - \tau h_2(\alpha)\,,$$

  where $\tau \in [\Sigma B \wedge \Sigma B, \Sigma B \wedge \Sigma B]$ is the map given by permuting the factors $\Sigma B$ in $\Sigma B \wedge \Sigma B$ (see [B-S, 3.17]).

For dimensional reasons, the homomorphisms

$$[\Sigma M/\partial M, \Sigma P/\partial P] \to \{M/\partial M, P/\partial P\}$$

and

$$[\Sigma^2 M/\partial M, (\Sigma P/\partial P)^{[2]}] \to \{M/\partial M, D_2(P/\partial P)\}$$

are onto (we are just one dimension out of the stable range). Hence, in this instance the identities for $h_2$ (using the compatibility identity) descend correspondingly to analogous identities for $H_2$.

The first identity for $\mu$ is then a direct consequence of the Cartan formula for $H_2$ (which is induced by the Cartan formula for $h_2$).

As for the second identity for $\mu$, Since $P/\partial P$ is a suspension, the transfer identity shows that

$$\alpha \cdot \alpha \quad = \quad H_2(\alpha) + \tau H_2(\alpha)$$

where $\tau\colon D_2(B) \longrightarrow D_2(B)$ is the interchange map. (Note the sign change: this is because the interchange map $(\Sigma B)^{[2]} \to (\Sigma B)^{[2]}$ covers the map $\Sigma^2 D_2(B) \to \Sigma^2 D_2(B)$ which not only permutes the factors in the $D_2(B)$ term, but also the suspension coordinates.)

Moreover, we have $\tau H_2(\alpha) = (-1)^p H_2(\alpha)$ since $\tau\colon D_2(P/\partial P) \longrightarrow D_2(P/\partial P)$ induces multiplication by $(-1)^p$ in $(2p)$-dimensional homology.

Consequently,

$$\alpha \cdot \alpha \quad = \quad H_2(\alpha) + (-1)^p H_2(\alpha)\,.$$

This induces second identity for $\mu$.

As for the third identity for $\mu$, in view of the other two identities it is sufficient to show that $\mu(-x) = \mu(x)$, since we can use the Cartan and trace formulas to expand $\mu(ax) = \mu((a-1)x + x)$ and then apply induction. Note $(-1)^p\mu(x) = \mu(x)$, since when $p$ is odd the equation lives in $\mathbb{Z}/2$. But the statement $\mu(-x) = (-1)^p\mu(x)$ is clearly a consequence of the identity $H_2(-\alpha) = (-1)^p H_2(\alpha)$ which we just derived. □

*Remark* 5.6. The above proof used well-known identities for the unstable Hopf invariant, which were shown to induce analogous identities for the stable Hopf invariant one dimension outside the stable range. Presumably, the identities for the stable Hopf invariant hold in general, but we could not find a reference.

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