POINCARÉ IMMERSIONS

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Abstract. We show that the fundamental theorem of immersion theory admits a Poincaré duality space analogue. Along the way, we obtain new homotopy theoretic proofs of the existence and uniqueness of the Spivak normal fibration of a closed Poincaré space.

1. Introduction

Assume that V^n and N^n are smooth n-manifolds, possibly with boundary and with V compact. Let $g\colon V \to N$ be a map. Consider the statements:

1. The map $g\times \mathrm{id}_{D^j}\colon V\times D^j \to N\times D^j$ is homotopic to a smooth embedding for some $j \geq 0$.
2. There exists a stable vector bundle isomorphism $\nu_V \cong g^*\nu_N$, where ν_N and ν_V are the stable normal bundles.

The implication $1 \Rightarrow 2$ is shown to hold by taking the differential of an embedding in the homotopy class of $g\times\mathrm{id}_{D^j}$. The other direction, $2 \Rightarrow 1$, can be proved as follows: a choice of bundle isomorphism gives rise to a tangent bundle monomorphism $T(V\times D^j) \to T(N\times D^j)$ covering $g\times\mathrm{id}_{D^j}$, whenever j is sufficiently large. The Hirsch immersion theorem [Hi] produces a smooth immersion in the homotopy class of $g\times\mathrm{id}_{D^j}$. Transversality enables one to perturb this immersion to an embedding if j is large.

Poincaré immersions. It is natural to ask whether a result of this kind holds in the category of Poincaré duality spaces. Let P^n and M^n be connected Poincaré spaces of dimension n (with or without boundary). In analogy with the above, and by a slight misuse of terminology, let us say that a map $f\colon P \to M$ *immerses* if $f\times\mathrm{id}_{D^j}\colon P\times D^j \to M\times D^j$ is the underlying map of a Poincaré embedding for some $j \geq 0$ (for the definition of Poincaré embedding see §2). We call a representative embedding an *immersion* of f. In this context we prove (see 3.1)

Date: November 12, 1998.

Theorem A. *(1). An immersion of $f\colon P^n \to M^n$ induces a stable fiber homotopy equivalence $\nu_P \simeq f^*\nu_M$ where ν_P and ν_M are the Spivak normal fibrations of P and M respectively.*

(2). Conversely, a map $f\colon P \to M$ and a stable fiber homotopy equivalence $\nu_P \simeq f^*\nu_M$ determine an immersion of f.

For certain applications, it is desirable to know the extent to which the fiber homotopy equivalence given Theorem A(1) is compatible with the immersion given in Theorem A(2).

Addendum. *Let $\phi\colon \nu_P \simeq f^*\nu_M$ be a fiber homotopy equivalence. Then the immersion in Theorem A(2) can be chosen so that its associated fiber homotopy equivalence $\nu_P \simeq f^*\nu_M$ given by Theorem A(1) coincides with ϕ up to fiber homotopy.*

Spivak thickenings. The main idea in proving Theorem A(2) is the notion of *Spivak thickening*, a homotopy theoretic version of the boundary of a regular neighborhood of a finite polyhedron embedded in euclidean space.

Let K be a connected homotopy finite space. An *n-thickening* of K is a homotopy finite space A and a map $i\colon A \to K$ such that the mapping cylinder pair $(K \cup_i A\times I, A\times 1)$ has the structure of an n-dimensional Poincaré space, and moreover, i is $(n-k-1)$-connected whenever K is homotopy equivalent to a CW complex of dimension $\leq k$. An n-thickening is a *Spivak n-thickening* if it comes equipped with a based map $\alpha\colon S^n \to K \cup_i CA$, such that $\alpha_*([S^n])$ is a fundamental class, and $K \cup_i CA$ is the mapping cone of i. The *decompression* of a Spivak n-thickening (A, α) is the Spivak $(n+1)$-thickening $(\Sigma_K A, \Sigma\alpha)$ where $\Sigma_K A = K\times 0 \cup A\times I \cup K\times 1$ is the fiberwise suspension of $A \to K$ (note: the mapping cone of $\Sigma_K A \to K$ is identified with the reduced suspension of the mapping cone of $A \to K$). Two Spivak n-thickenings (A, α) and (A', α') are *elementary concordant* if there exists a weak homotopy equivalence $h\colon A \xrightarrow{\sim} A'$ covering the identity of K such that the induced map of mapping cones $K \cup CA \to K \cup CA'$ transfers α to α'. *Stable concordance* of Spivak thickenings is the equivalence relation generated by decompression and elementary concordance.

Theorem B. *A connected homotopy finite space possesses a Spivak thickening which is unique up to stable concordance.*

(In fact, stable concordances between any two Spivak thickenings can be chosen in some sense canonically, in a way which we won't try to make precise.)

We will give two proofs of Theorem B. The first uses manifold techniques: regular neighborhood theory and the Browder-Casson-Sullivan-Wall theorem. The second proof uses equivariant duality with respect to a topological group model for the loop space of K. The second proof is manifold-free.

An important special case occurs when K is a closed (= without boundary) Poincaré space of dimension k. If $k \leq n-3$ it can be shown that the homotopy fiber of the map $A \to K$ underlying an n-thickening is homotopy equivalent to an $(n-k-1)$-sphere (see [Br2, I.4.4] or [Kl1, Th. B]). Theorem B immediately gives the classical existence and uniqueness results for the Spivak normal fibration:

Corollary C (Spivak [Sp], Wall [Wa2]). *Assume that K^k is a closed Poincaré space. Then there exist an integer $n \geq k$, an $(n-k-1)$-spherical fibration $\nu\colon S(\nu) \to K$ and a map $\alpha\colon S^n \to K^\nu$ such that the cap product $U \cap \alpha_*([S^n])$ is a fundamental class for K, where K^ν denotes the Thom space of ν, and U denotes a Thom class for ν.*

When $n \gg k$ is large, any two such pairs (E, α) and (E', α') are related by a stable fiber homotopy equivalence $E \xrightarrow{\simeq} E'$ such that the induced map of Thom spaces transfers α to α' up to homotopy.

The classical proof of Corollary C invokes regular neighborhood theory for existence and Spanier-Whitehead duality for uniqueness. Our two proofs of Theorem B yield two proofs of Corollary C. The first one more-or-less amounts to the classical proof, but the second one seems to be new, and is based on the existence of Equivariant Spanier-Whitehead duality maps. Bill Dwyer has informed me that he was also aware of an argument of this kind. A rough sketch of the second argument yielding the existence part of Corollary C in the simply connected case appears in [Kl2].

Outline. The material in §2 is for the most part language. In §3 we prove the first part of Theorem A and provide a quick proof of the second part when the source also admits the structure of a closed Poincaré space of dimension $p \leq n$. In §4 we give the manifold proof of Theorem B. In §5 we prove the second part Theorem A as a consequence of Theorem B. In §6 we develop equivariant duality. In §7 we provide a second proof of Theorem B using the material in §6. In §8 we prove the addendum to Theorem A.

Acknowledgements. I would like to thank Andrew Ranicki for bringing these issues to my attention.

2. Preliminaries

Spaces. Our ground category is **Top**, the category of compactly generated Hausdorff spaces, equipped with its usual Quillen model category structure (weak homotopy equivalences, Serre fibrations and Serre cofibrations; see [Qu, II.3]).

A non-empty space is always (-1)-connected. A connected space is 0-connected, and is r-connected for some $r > 0$ if its homotopy groups vanish up through degree r, for any choice of basepoint. A map of non-empty spaces $X \to Y$ is called *r-connected* if its homotopy fiber with respect to any choice of basepoint in Y is an $(r-1)$-connected space. An ∞-connected map is a weak equivalence.

A space is *homotopy finite* if it is homotopy equivalent to a finite CW complex.

If $Y \leftarrow X \to Z$ is a diagram of cofibrant spaces, its *homotopy pushout* is the double mapping cylinder $Y\times 0 \cup X\times[0,1] \cup Z\times 1$. If $Y \to X$ is a map of cofibrant spaces, then the homotopy pushout of $X \leftarrow Y \to X$ is the *(unreduced) fiberwise suspension* $\Sigma_X Y$ of $Y \to X$; it comes equipped with an evident map $\Sigma_X Y \to X$.

If $Y \to X$ is a map of cofibrant spaces, we will often write $(\bar{X}, Y)$ for the cofibration pair $(X \cup_{Y\times 0} Y\times[0,1], Y\times 1)$ associated with the mapping cylinder.

A commutative diagram of spaces

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array}$$

is *cocartesian* (née homotopy cocartesian) if the induced map from the homotopy pushout of $B \leftarrow A \to C$ to D is a weak equivalence.

Finally, a note about usage. In the majority of instances below, the term 'space' will refer to a cofibrant object of **Top**, i.e., the retracts of objects built up from the empty space by attaching a finite number of cells. However, there is one notable exception: in dealing with spherical fibrations $\xi\colon S(\xi) \to X$, we will *not* always require the total space $S(\xi)$ to be cofibrant. The reason for this is that we will be forming base changes $S(\xi)_{|Y} \to Y$ along maps $Y \to X$ (here $S(\xi)_{|Y} = Y\times_X S(\xi)$ is one of the notations we will be using for the fiber product); cofibrancy of total spaces is not usually invariant under base changes.

Fibrations. Assume B is a cofibrant space. We let $\mathbf{Top}/B$ denote the category of spaces over B: an *object* consists of a map $Y \to B$, where Y is an object of **Top**. A *morphism* $Y \to Z$ is a map covering the identity map of B. Quillen [Qu, II.2.8] proved that $\mathbf{Top}/B$ has the structure of a model category in which the weak equivalences and fibrations are defined by applying the forgetful functor to **Top**. A fibration $E \to B$ of spaces is the same thing as fibrant and cofibrant object of $\mathbf{Top}/B$. Given fibrant and cofibrant objects $E, E' \in \mathbf{Top}/B$, there exists a fiberwise homotopy equivalence $E \simeq E'$ if and only if E and E' descend to isomorphic objects of the homotopy category of $\mathbf{Top}/B$ (here we are applying a well-known fact about model categories which says that a pair of fibrant and cofibrant objects are homotopy equivalent if and only are related by a chain of weak equivalences).

Incidentally, we do *not* assert (since we do not know) that the fiberwise suspension $\Sigma_B E \in \mathbf{Top}/B$ of a fibrant and cofibrant object is again fibrant. It is however cofibrant. Given fibrant objects $E, E' \in \mathbf{Top}/B$, their *fiberwise join* $E *_B E' \in \mathbf{Top}/B$ is the (cofibrant) object given by the double mapping cylinder of the diagram $E^{\mathrm{c}} \leftarrow (E \times_B E')^{\mathrm{c}} \to (E')^{\mathrm{c}}$ (again, we do not assert that $E *_B E'$ is fibrant).

If B is a connected space, define Sph_B to be the set generated by objects $E \to B$ whose homotopy fiber is spherical, modulo the equivalence relation generated by weak equivalence in $\mathbf{Top}/B$ and fiberwise suspension. Fiberwise join equips Sph_B with the structure of an abelian monoid. If additionally B is homotopy finite, then a result of Stasheff [St] implies Sph_B is an abelian group.

Thom spaces. Let X be a space equipped with a spherical fibration $\xi\colon S(\xi) \to X$. The *Thom space* X^ξ is the mapping cone of the composite $S(\xi)^{\mathrm{c}} \xrightarrow{\sim} S(\xi) \to X$, where $S(\xi)^{\mathrm{c}}$ denotes the cofibrant approximation of $S(\xi)$. Then X^ξ is cofibrant in the sense of based spaces.

Poincaré spaces. A *Poincaré space* X of dimension n is a pair $(X, \partial X)$ such that X and ∂X are homotopy finite, $\partial X \to X$ is a cofibration, and X satisfies *Poincaré duality:*

- there exist a local system of abelian groups $\mathcal{L}$ of rank one defined on X and a fundamental class $[X] \in H_n(X, \partial X; \mathcal{L})$ such that the cap product homomorphisms

$$\cap[X]\colon H^*(X; M) \to H_{n-*}(X, \partial X; \mathcal{L} \otimes M)$$

and

$$\cap[\partial X]\colon H^*(\partial X; N) \to H_{n-*-1}(\partial X; \mathcal{L}_{|\partial X} \otimes N)$$

are isomorphisms, where $[\partial X] \in H_{n-1}(\partial X; \mathcal{L}_{|\partial X})$ is the image of $[X]$ under the connecting homomorphism in the homology exact sequence of the pair $(X, \partial X)$, and M (N) is any local system on X (resp. on ∂X) (compare [Wa2]).

A *Spivak fibration* $\nu\colon S(\nu) \to X$ is an oriented spherical fibration which comes equipped with a based map $\alpha\colon S^{n+j} \to X^\nu/(\partial X)^{\nu_{|\partial X}}$ (where, say, the fiber of ν is $(j-1)$-spherical) such that the cap product

$$U \cap \alpha_*([S^{n+j}])$$

is a fundamental class for X. Here $U \in H^j(\nu)$ denotes a Thom class of ν, where $H^j(\nu)$ is the j-th singular cohomology group of the pair $(\bar{X}, S(\nu)^{\mathrm{c}})$ with coefficients taken in the local system $\mathcal{L}$. The stable fiber homotopy type of ν is unique, more precisely, two such pairs (ν, α) and (ν', α') can be related by a stable fiber homotopy equivalence $\nu \simeq \nu'$ whose induced map on Thom spaces transfers α onto α' up to homotopy (see [Wa2, 3.4-5] and [Br2, I.4.19]; alternatively we will recover the existence and uniqueness of the Spivak fibration in Corollary C).

Embeddings. Let P^n and M^n be connected Poincare spaces of dimension n. An *embedding* of P in M is a commutative cocartesian square of homotopy finite spaces

$$\begin{array}{ccc} \partial P & \longrightarrow & C \\ {\scriptstyle \text{incl.}}\downarrow & & \downarrow{\scriptstyle g} \\ P & \underset{e}{\longrightarrow} & M \end{array}$$

together with a factorization of the inclusion $\partial M \to C \to M$, such that the composite

$$H_n(M, \partial M) \to H_n(\bar{M}, C) \cong H_n(P, \partial P)$$

is an isomorphism and the image of a fundamental class $[M]$ under the composite

$$H_n(M, \partial M) \to H_n(\bar{M}, P \amalg \partial M) \cong H_n(\bar{C}, \partial P \amalg \partial M)$$

equips $(\bar{C}, \partial P \amalg \partial M)$ with the structure of a Poincaré space

The space C is called the *complement*, and $e\colon P \to M$ is the *underlying map* of the embedding. In this case, we say that the map $e\colon P \to M$ *embeds*. Sometimes we will have occasion to refer to the embedding by means of its underlying map.

The *decompression* of an embedding $e\colon P \to M$ with complement C is the embedding $P\times I \to M\times I$ defined by the diagram

$$\begin{array}{ccc} \partial(P\times I) & \longrightarrow & W \\ \downarrow & & \downarrow \\ P\times I & \xrightarrow[e\times \mathrm{id}_I]{} & M\times I \end{array}$$

where $W := \Sigma_M C$, and the factorization $\partial(P\times I) \to W \to M\times I$ is evident (since $\Sigma_P \partial P \cong \partial(P\times I)$).

Let K be a connected homotopy finite space, X^n be a Poincaré space and $f\colon K \to X$ a map. One says that f *embedded thickens* (née PD embeds [Kl3]) if there exists a map $A \to K$ ('normal data') such that $(\bar{K}, A)$ is an n-dimensional Poincaré pair, and such that with respect to this choice, the composite

$$\bar{K} \xrightarrow{\simeq} K \xrightarrow{f} X$$

embeds (in the above sense). It was shown in [Kl3, 3.1] that the composite

$$K \xrightarrow{f} X \xrightarrow{\subset} X\times D^j$$

embedded thickens if j is sufficiently large.

3. Immersions

As in the introduction, a map $f\colon P^n \to M^n$ of connected n-dimensional Poincaré spaces *immerses* if $f\times \mathrm{id}_{D^j}\colon P\times D^j \to M\times D^j$ embeds for some integer $j\geq 0$; such an embedding is called an *immersion* of f. We now restate Theorem A.

Theorem 3.1. *Let $f\colon P^n \to M^n$ be a map. Then*

1. *An immersion of f gives rise to a stable fiber homotopy equivalence $\phi\colon \nu_P \simeq f^*\nu_M$.*
2. *Conversely, given a stable fiber homotopy equivalence $\phi\colon \nu_P \simeq f^*\nu_M$, there is an immersion of f which induces it up to stable fiber homotopy.*

Proof of 3.1(1). Let (ν, α) be a Spivak fibration for M as guaranteed by [Sp, Th. A] or by Corollary C, where say, $\nu\colon S(\nu) \to M$ and $\alpha\colon S^{n+j} \to M^\nu/(\partial M)^{\nu_{|\partial M}})$.

Suppose first that f embeds. Let

$$\begin{array}{ccc} \partial P & \longrightarrow & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ P & \underset{f}{\longrightarrow} & M \end{array}$$

be an embedding diagram. Consider the composite β, defined by

$$S^{n+j} \xrightarrow{\alpha} M^{\nu}/(\partial M)^{\nu_{|\partial M}} \to M^{\nu}/C^{j^*\nu} \simeq P^{f^*\nu}/(\partial P)^{(f\circ i)^*\nu}$$

(to avoid notational clutter we are fudging slightly in writing $M^\nu/C^{j^*\nu}$ for the mapping cone of the map $C^{j^*\nu} \to M^\nu$).

If $U_{f^*\nu}$ denotes a Thom class of $f^*\nu$, then by the naturality of the Thom isomorphism the cap product $U_{f^*\nu} \cap \beta_*([S^{n+j}])$ is a fundamental class for P. We infer that $f^*\nu$ is a Spivak normal fibration for P. By the uniqueness of the Spivak fibration ([Wa2, 3.5], [Br2, I.4.19] or Corollary C), there is a stable fiber homotopy equivalence $\nu_P \simeq f^*\nu$. The establishes 3.1(1) for embeddings of f.

The general case of 3.1(1) follows by replacing the above embedding of f by an embedding of $f \times \mathrm{id}_{D^j} \colon P \times D^j \to M \times D^j$. □

We next give a quick proof of 3.1(2) in the special case when the underlying space of P also has the structure of a *closed* Poincaré space. The argument for the general case is in §5.

Proof of 3.1(2) when P is a closed Poincaré space. Assume P^n has the structure of a p-dimensional closed Poincaré space ($p \le n$). To distinguish the two Poincaré duality structures, we let V^p denote the effect of forgetting the boundary of P. Let ν_V be a Spivak fibration of V. Then the map $\partial P \to P = V$ has an $(n-p-1)$-spherical homotopy fiber. Let ∂ be the result of making $\partial P \to P$ into a fibration. Then $\nu_P + \partial = \nu_V$ in Sph_P. By hypothesis $f^*\nu_X \simeq \nu_P$, so

$$\nu_V = f^*\nu_X + \partial\,.$$

in Sph_P.

Now choose an embedded thickening of the composite $V \xrightarrow{f} X \subset X \times D^j$; this can be achieved when j is chosen sufficiently large (see [Kl3, 3.1]). Then $\nu_V = f^*\nu_X + \xi$ in Sph_P, where $\xi \colon A \to V$ is the normal data of the embedded thickening. We infer that $f^*\nu_X + \partial = f^*\nu_X + \xi$, whence $\partial = \xi$ in Sph_P. Consequently, the embedded thickening amounts to an embedding of $f \colon P \times D^j \to X \times D^j$. □

4. First Proof of Theorem B

The argument which proves the general case of 3.1(2) employs Poincaré thickenings. Recall the statement of Theorem B:

Theorem 4.1. *Given a connected homotopy finite space K, there is one and only one Spivak thickening of K up to stable concordance.*

As stated in the introduction, we shall provide two proofs of 4.1. The first proof uses manifolds and appears in this section. The second proof is homotopy theoretic and will appear in §7.

First proof of 4.1. The existence of a Spivak thickening of K is given as follows: K may replaced up to homotopy by a finite k-dimensional polyhedron embedded in some Euclidean space. The boundary of the regular neighborhood defines a Poincaré boundary A, the retraction of the regular neighborhood defines an $(n-k-1)$-connected map $A \to K$, and the Thom-Pontryagin collapse defines a map $S^n \to \bar{K}/A$ representing a fundamental class.

To prove uniqueness we introduce an auxiliary notion: an *embedded smooth thickening* of K in S^n is a compact codimension zero smooth submanifold $N^n \subset S^n$ and a homotopy equivalence $h\colon K \xrightarrow{\simeq} N$.[1] Two embedded smooth thickenings (N_0, h_0) and (N_1, h_1) of K are *concordant* if there exist an embedded h-cobordism $W \subset S^n \times I$ from $N_0 \subset S^n \times 0$ to $N_1 \subset S^n \times 1$ and a homotopy equivalence $H\colon K \times I \xrightarrow{\simeq} W$ extending h_0 and h_1. Regular neighborhood theory and transversality show that there is one and only one embedded smooth thickening of K in S^n up to concordance when n is large with respect to $k :=$ the minimum dimension of those CW complexes in the homotopy type of K.

Let (A, α) and (A', α') be Spivak n-thickenings of K. By a construction due to Browder [Br1], the pair (A, α) defines an embedded (Poincaré) thickening in S^{n+1}:

$$\begin{array}{ccc} \Sigma_K A & \longrightarrow & C \\ \big\downarrow & & \big\downarrow \\ K & \longrightarrow & S^{n+1} \end{array}$$

where the complement C is the mapping cone of $\alpha\colon S^n \to \bar{K}/A$ and the map $\Sigma_K A \to C$ is given by collapsing the inclusion of $K \times 0$ in $\Sigma_K A = K \times 0 \cup A \times [0,1] \cup K \times 1$ to a point. In what follows, we let

[1] This notion is sometimes called an *embedding up to homotopy type* and is similar to the definition of Wall [Wa1], the essential difference being that Wall requires h to be a simple homotopy equivalence.

Σ^{n+1} denote the double mapping cylinder of $K \leftarrow \mathit{\Sigma}_K A \to C$. This gives a triad of spaces $(\Sigma^{n+1}; K, C; \mathit{\Sigma}_K A)$.

If $n \geq 5$, the Browder-Casson-Sullivan-Wall theorem [Wa3, 12.1] implies that there exist an embedded smooth thickening (N^n, h) of K in S^{n+1}, with complement W and a homotopy equivalence of triads

$$(S^{n+1}; N, W; \partial N) \xrightarrow{\simeq} (\Sigma; K, C; \mathit{\Sigma}_K A)\,.$$

This in turn defines a concordance between the Spivak $(n{+}1)$-thickenings $(\mathit{\Sigma}_K A, \alpha)$ and $(\partial N, a)$, where $a\colon S^n \to N/\partial N$ denotes the Thom-Pontryagin collapse map of $N \subset S^n$.

Repeating the above procedure with (A, α) replaced by (A', α'), we obtain another embedded smooth thickening (N', h') of K in S^{n+1} and a homotopy equivalence of triads

$$(S^{n+1}; N', W'; \partial N') \xrightarrow{\simeq} (\Sigma'; K, C'; \mathit{\Sigma}_K A')\,.$$

Therefore we have a concordance between the Spivak $(n{+}1)$-thickenings $(\mathit{\Sigma}_K A', \alpha')$ and $(\partial N', a')$, where a' is the Thom-Pontryagin collapse of $N' \subset S^{n+1}$.

Assuming n is large, and using the stable uniqueness up to concordance of embedded smooth thickenings, we see that (N, h) and (N', h') are concordant. This concordance induces, in an evident way, a concordance between $(\partial N, a)$ and $(\partial N', a')$. It follows that $(\mathit{\Sigma}_K A, \mathit{\Sigma}\alpha)$ and $(\mathit{\Sigma}_K A', \mathit{\Sigma}\alpha')$ are concordant. □

5. Completion of the proof of Theorem A

Using Theorem B, we give the proof of 3.1(2).

Construction 5.1. Let K be a connected space, $\xi\colon S(\xi) \to K$ a spherical fibration and $E \to K$ a map of spaces. Let $\mathit{\Sigma}^\xi_K E$ be the double mapping cylinder of

$$S(\xi)^{\mathrm{c}} \leftarrow (S(\xi)_{|E})^{\mathrm{c}} \to E$$

(this is a kind of twisted fiberwise suspension of $E \to K$). There is an evident map $\mathit{\Sigma}^\xi_K E \to K$. Furthermore, there is a cocartesian square

$$\begin{array}{ccc} \mathit{\Sigma}^\xi_K E & \longrightarrow & E^{\xi_{|E}} \\ \big\downarrow & & \big\downarrow \\ K & \longrightarrow & K^\xi \end{array}$$

which shows that the mapping cone of $\mathit{\Sigma}^\xi_K E \to K$ coincides up to homotopy with the mapping cone of $E^{\xi_{|E}} \to K^\xi$. If $\epsilon_j\colon K \times S^{j-1} \to K$

denotes the trivial $(j-1)$-spherical fibration, then $\Sigma_K^{\epsilon_j} E$ is identified with the j-fold iterated fiberwise suspension $\Sigma_K^j E$.

If $E \to K$ is an n-thickening of K and if ξ is $(j-1)$-spherical, then $\Sigma_K^\xi E \to K$ is an $(n+j)$-thickening of K.

Proof of 3.1(2). Set $\nu = \nu_M$. Given a map of spaces $g\colon Y \to M$, to avoid notational clutter we write Y^ν for the Thom space $Y^{g^*\nu}$. By assumption, $f^*\nu_M \simeq \nu_P$, so we have an identification $P^{\nu_P} = P^\nu$. Using the construction 5.1, we have a thickening $\Sigma_P^\nu \partial P \to P$ which has $P^\nu/(\partial P)^\nu$ as its mapping cone (up to homotopy). Hence, $\Sigma_P^\nu \partial P \to P$ is a Spivak thickening.

On the other hand, we also know that the composite $P \xrightarrow{f} M \subset M \times D^j$ embedded thickens for large j (by [Kl3, 3.1]). Choose such an embedded thickening, and let $A \to P$ be its normal data. Then $\Sigma_P^\nu A \to P$ is also a Spivak thickening of P. Reason: let W denote the complement of the embedded thickening. Then there is a homotopy equivalence $P^\nu/A^\nu \simeq M^\nu/W^\nu$ since A, P, W and M sit in a cocartesian square (here and in what follows, in order to avoid notational clutter, we are fudging slightly in writing P^ν/A^ν for the mapping cone of $A^\nu \to P^\nu$). Let $S^\ell \to M^\nu/(\partial M)^\nu$ represent a fundamental class of $(\bar{M}, \Sigma_M^\nu \partial M)$. Follow this up with the map $M^\nu/(\partial M)^\nu \to M^\nu/W^\nu \simeq P^\nu/A^\nu$ to obtain a map $S^\ell \to P^\nu/A^\nu$ representing a fundamental class of $(\bar{P}, \Sigma_P^\nu A)$. This shows that $\Sigma_P^\nu A \to P$ is a Spivak thickening (again, we are using the identification $P/\Sigma_P^\nu A \simeq P^\nu/A^\nu$). Consequently, by 4.1, $\Sigma_P^\nu A \to P$ and $\Sigma_P^\nu \partial P \to P$ are stably concordant.

Let τ represent an inverse for ν_P in Sph_P (cf. §2). Then, stabilizing further with respect to τ, it follows that $\Sigma_P^{\nu+\tau} A$ and $\Sigma_P^{\nu+\tau} \partial P$ are isomorphic in the stable homotopy category of spaces over P. Equivalently, there exist non-negative integers j and ℓ such that $\Sigma_P^j A$ and $\Sigma_P^\ell \partial P$ are isomorphic in the homotopy category of spaces over P. But $\Sigma_P^j A \to P$ amounts to the normal data for the j-fold decompression of the given embedded thickening of $f\colon P \to M$ with normal data $A \to P$. We infer that $P \xrightarrow{f} M \subset M \times D^\ell$ embedded thickens with normal data $\Sigma_P^\ell \partial P \to P$. But the pair $(\bar{P}, \Sigma_P^\ell \partial P)$ is identified with $(P \times D^\ell, \partial(P \times D^\ell))$, so what we have is an embedding whose underlying map is $f \times \mathrm{id}_{D^\ell}\colon P \times D^\ell \to M \times D^\ell$. We conclude that $f\colon P \to M$ immerses. □

6. Equivariant duality

We develop duality for spaces equipped with the action of a suitable kind of topological group G with coefficients in the so-called naive G-spectra. The reader should not confuse our theory with the equivariant duality appearing in [B-M-S-M, Chap. 3], rather, our theory reduces to Ranicki duality [Ra, §3] in the case when G is discrete.

Let $G.$ be a simplicial group. Then $G = |G.|$ (i.e., the realization of its underlying simplicial set) is a topological group object in **Top**. Let $R^G(*)$ denote the category of based (left) G-spaces (where 'space' in this context means an object of **Top**) and equivariant based maps. Declare a morphism of $R^G(*)$ to be a *weak equivalence* (resp. *fibration*) if and only if it is one after applying the forgetful functor $R^G(*) \to$ **Top**. A morphism is a *cofibration* if and only if it satisfies the left lifting property with respect to the fibrations which are also weak equivalences. Then

Proposition 6.1. *With respect to the above, $R^G(*)$ is a Quillen model category.*

For a proof, see [S-V].

Since we will be for the most part be using cofibrant objects, we indicate what these are. The equivariant j-cell is the *un*based G-space $D^j \times G$ where G-acts by left translation. The equivariant $(j-1)$-sphere is similarly $S^{j-1} \times G$. If $Z \in R^G(*)$ is an object and $g\colon S^{j-1} \times G \to Z$ is a G-map, then we may form the amalgamation $Z \cup_g (D^j \times G)$, which is again an object of $R^G(*)$. Call an object of $R^G(*)$ *cofibrant* if it is built up from a point by iterated equivariant cell-attachments, or if it is a retract thereof (in particular, a cofibrant object is free away from the basepoint). Any object Z comes equipped with a functorial cofibrant approximation $Z^{\mathrm{c}} \xrightarrow{\sim} Z$.

An object of $R^G(*)$ is *finite* if it is isomorphic to an object built up from a point by a finite number of equivariant cell attachments. It is *homotopy finite* if it admits a chain of weak equivalences to a finite object (i.e., they are isomorphic in the homotopy category).

The reduced suspension ΣY of a cofibrant object Y (formed as in the category of based spaces) is again a cofibrant object; the action of G on ΣY is defined by letting G act trivially on the suspension coordinate. For cofibrant objects Y, Z we define $[Y, Z]^G$ to be equivariant homotopy classes of equivariant maps.

Given cofibrant objects $Y, Z \in R^G(*)$, the smash product $Y \wedge Z$ (formed as in the category of based spaces) inherits the diagonal action of G. Let $Y \wedge_G Z$ denote the effect of taking orbits with respect to the action of G. Then $Y \wedge_G Z$ is an (unequivariant) based space.

A *spectrum with G-action E* consists of cofibrant objects $E_i \in R^*(G)$ for integers $i \geq 0$ and morphisms $\Sigma E_i \to E_{i+1}$. (We remark that these are equivariant spectra in the naive sense, and are not to be confused with the more elaborate notion used by May *et. al.* [B-M-S-M].)

Given a cofibrant object $Y \in R^G(*)$, the *equivariant cohomology* of Y with coefficients in E is the graded abelian group

$$E^*_G(Y) \quad := \quad \operatorname*{colim}_{j\to\infty} [\Sigma^j Y, E_{j+*}]^G .$$

Similarly, *equivariant homology* of Y with coefficients in E is the graded abelian group

$$E^G_*(Y) \quad := \quad \operatorname*{colim}_{j\to\infty} [S^{j+*}, E_j \wedge_G Y] .$$

Definition 6.2. Assume that $Y, Z \in R^G(*)$ are cofibrant and homotopy finite objects. Let $d\colon S^n \to Y \wedge_G Z$ be a map. Then d is an *equivariant duality with respect to E* if the correspondence

$$\Sigma^j Y \xrightarrow{f} E_{j+*} \quad \longmapsto \quad S^{n+j} \xrightarrow{(f\wedge_G \mathrm{id}_Z)\circ d} E_{j+*} \wedge_G Z$$

induces an isomorphism of abelian groups

$$E^*_G(Y) \xrightarrow{\cong} E^G_{*-n}(Y) .$$

If equivariant duality holds with respect to all spectra with G-action E, then d is said to be an *equivariant duality map.*

If $d\colon S^n \to Y \wedge_G Z$ is an equivariant duality map, then so is its suspension $\Sigma d\colon S^{n+1} = \Sigma S^n \to \Sigma(Y \wedge_G Z) = Y \wedge_G \Sigma Z$.

Example 6.3. Let $i \leq n$ be a nonnegative integer. Set

$$S^n_G \ := \ S^n \wedge (G_+)$$

where G_+ means G with the disjoint union of a basepoint, and where G acts on S^n_G by left translation. The diagonal map $G_+ \to (G\times G)_+$ smashed with $S^n = S^i \wedge S^{n-i}$ defines an equivariant map $S^n_G \to S^i_G \wedge S^{n-i}_G$. Taking G-orbits gives a map $S^n \to S^i_G \wedge_G S^{n-i}_G$. It is straightforward to check that the latter is an equivariant duality map.

To determine whether a map is an equivariant duality, it is sufficient to verify duality with respect to a suitable version of equivariant singular homology. We now explain what this means.

Let $\pi := \pi_0(G)$ be the group of path components of G. Then $G \to \pi$ is a homomorphism. Let G_0 denote the kernel. The diagonal inclusion $G_0 \subset G_0 \times G_0$ defines a map

$$Y \wedge_{G_0} Z \to Y_{G_0} \wedge Z_{G_0} ,$$

where Y_{G_0} denotes the based π-space given by taking the orbits of G_0 acting on Y. This map is π-equivariant, so we may take π-orbits to obtain a canonical map $Y \wedge_G Z \to Y_{G_0} \wedge_\pi Z_{G_0}$.

Proposition 6.4. *A map $d\colon S^n \to Y \wedge_G Z$ is an equivariant duality if and only if taking the slant product with the fundamental class of S^n with respect to the composite*

$$d_0\colon S^n \xrightarrow{d} Y \wedge_G Z \to Y_{G_0} \wedge_\pi Z_{G_0}$$

yields an isomorphism

$$/d_{0*}([S^n])\colon H^*(Y_{G_0}) \xrightarrow{\cong} H_{n-*}(Z_{G_0})\,,$$

where H^* denotes reduced singular homology.

In other words, the map $d_0\colon S^n \to Y_{G_0} \wedge_\pi Z_{G_0}$ is an π-equivariant duality map with respect to $h(\mathbb{Z}[\pi])$, the Eilenberg-MacLane spectrum with π-action associated to integral group ring of π (cf. Ranicki [Ra, §3]). In the special case when π is trivial, this just means that d_0 is a Spanier-Whitehead duality map.

Proof of 6.4. We shall only provide a sketch of the argument (details will appear elsewhere). First we introduce some auxiliary definitions.

Define the *homotopy groups* of a spectrum with G-action E to be the graded $\mathbb{Z}[\pi]$-module $\pi_*(E) := \mathrm{colim}_j[S^{j+*}, E_j]$. A *morphism* of spectra with G-action $D \to E$ is a collection of morphisms $D_i \to E_i$ compatible with the structure maps of D and E. A morphism is a *weak equivalence* if it induces an isomorphism on homotopy groups. If $D \to E$ is a weak equivalence, it induces isomorphisms $D^G_*(Y) \cong E^G_*(Y)$ and $D^*_G(Y) \cong E^*_G(Y)$ for all cofibrant objects $Y \in R^G(*)$.

For any $\mathbb{Z}[\pi]$-module M, let $h(M,j)$ be its associated Eilenberg-Mac Lane spectrum in degree j. Then $h(M,j)$ is a spectrum with π-action. The homomorphism $G \to \pi$ provides an action of G on the spaces $h(M,j)_k$, however we have to free up the action to ensure that $h(M,j)$ is a spectrum with G-action. This can be accomplished by replacing the k-th π-space $h(M,j)_k$ by the cofibrant G-space $EG \times h(M,j)_k$ (with G acting diagonally). Assume this has been done.

With respect to the hypotheses of the proposition, the π-cellular chain slant product map

$$/d_{0*}([S^n])\colon C^*(Y_{G_0}) \to C_{n-*}(Z_{G_0})$$

is a π-equivariant chain equivalence. Equivalently, duality is satisfied with respect to the spectrum with G-action $h(\mathbb{Z}[\pi], j)$ for all j.

Tensoring the above map with any $\mathbb{Z}[\pi]$-module M, and using the finiteness of Y and Z, it follows that the slant product map

$$/d_{0*}([S^n])\colon C^*(Y_{G_0};M) \to C_{n-*}(Z_{G_0};M)$$

(where chains now have coefficients in M) is also an equivariant chain homotopy equivalence (compare [Wa2, 1.1]). Equivalently, G-equivariant duality holds with respect to $h(M,j)$ for all j and all $\mathbb{Z}[\pi]$-modules M.

A spectrum with G-action E is said to be *Eilenberg-Mac Lane* of type (M,j) for a $\mathbb{Z}[\pi]$-module M and an integer j if $\pi_*(E)$ is trivial if $* \neq j$ and there is an isomorphism of $\mathbb{Z}[\pi]$-modules $\pi_j(E) \cong M$. By an obstruction theory argument which we omit, one shows that an Eilenberg-MacLane spectrum of type (M,j) admits a weak equivalence $E \to h(M,j)$. This gives equivariant duality for spectra with G-action which are Eilenberg-MacLane of type (M,j). By the procedure of killing homotopy groups, any spectrum with G-action E admits a Postnikov system, i.e., a sequence $\cdots \to P_kE \to P_{k-1}E \to \cdots$ for $k \in \mathbb{Z}$ and compatible maps $E \to P_kE$ such that the homotopy groups of E vanish in degrees $>k$ and $E \to P_kE$ induces a isomorphism on homotopy groups in degrees $\leq k$. Then the homotopy fiber of $P_kE \to P_{k-1}E$ is Eilenberg-Mac Lane of type $(\pi_k(E),k)$. Call E *bounded below* if $\pi_*(E) = 0$ whenever $*$ is sufficiently small. By induction up the stages of the Postnikov system using the five lemma, we obtain equivariant duality for bounded below spectra with G-action.

Any spectrum with G-action E admits a chain of weak equivalences to a spectrum with G-action of the form $\operatorname{colim}_j E^{(j)}$ in which $E^{(j)}$ is bounded below (we can define $E^{(j)}$ as the $(-j)$-connected cover of E, i.e., the homotopy fiber of the map $E \to P_{-j}E$). Equivariant duality for E then follows from equivariant duality for the $E^{(j)}$. □

The fundamental result about the existence and uniqueness of equivariant duality maps is

Theorem 6.5. (Existence). *If $Y \in R^G(*)$ is cofibrant and homotopy finite, then there exist an integer $n \geq 0$, a cofibrant and homotopy finite object $Z \in R^G(*)$ and an equivariant duality map $d\colon S^n \to Y \wedge_G Z$.*

(Uniqueness). *Assume Y comes equipped with two equivariant duality maps $d\colon S^n \to Y \wedge_G Z$ and $d'\colon S^n \to Y \wedge_G Z'$. Then there exist an integer $j \geq 0$ and an equivariant weak equivalence $h\colon \Sigma^j Z \to \Sigma^j Z'$ such that $(id_Y \wedge_G h) \circ \Sigma^j d$ is homotopic to $\Sigma^j d'$. Moreover, if j is large then h is unique up to equivariant homotopy.*

Proof. (Existence). Ranicki [Ra, 3.5] provides a proof of existence with $G = \pi$ is discrete and Y is a finite object (he uses induction on the equivariant cell decomposition; the inductive step uses the duality map provided in example 6.3). An inspection his proof adapts to the case of a general G when Y is finite. The case of a general homotopy finite Y follows by noting that a cofibrant homotopy finite object Y admits a weak equivalence $Y' \xrightarrow{\sim} Y$ where Y' is built up from a finite number of equivariant cells.

(Uniqueness). The proof will be based on equivariant obstruction theory. A morphism $A \to B$ of $R^G(*)$ will be called *r-connected* if it is so as a map of spaces. For a cofibrant object A, we write $\dim A \leq k$ if A is built up from a point by attaching equivariant cells of dimension $\leq k$. We will require the following fact, whose proof is given by induction on the number of equivariant cells of A: given an r-connected morphism $f\colon A \to B$ and a map $g\colon Z \to B$ with Z cofibrant and $\dim Z \leq r$, there exists a map $\tilde{g}\colon Z \to A$ such that $f \circ \tilde{g}$ is equivariantly homotopic to g. Moreover, $\tilde{g}$ is unique up to homotopy if $\dim Z \leq r{-}1$.

Assume that $d\colon S^n \to Y \wedge_G Z$ and $d'\colon S^n \to Y \wedge_G Z'$ are equivariant duality maps. We may assume without loss in generality that Y, Z and Z' are finite objects.

Let $F(Y, S^n_G)$ denote the function space of based (unequivariant) maps from Y to S^n_G (recall $S^n_G := S^n \wedge (G_+)$), topologized using the compactly generated compact open topology. Then $F(Y, S^n_G)$ is a based G-space where G acts on functions by conjugation. The fixed point space $F(Y, S^n_G)^G$ is the space of equivariant functions. This eqivariant function space admits a left G-action using the right translation action on S^n_G. The correspondence given by $f \mapsto (f \wedge_G \mathrm{id}) \circ d$ defines a map

$$\delta_Z\colon F(Y, S^n_G)^G \to F(S^n, S^n \wedge Z)$$

and if we give $F(S^n, S^n \wedge Z)$ the action induced by the action of G on Z and the trivial action on S^n, it follows that δ_Z is G-equivariant. Equivariant duality implies that δ_Z induces an isomorphism on homotopy groups in a certain range (the range where the function spaces homotopically coincide with their associated function spaces of stable maps). Specifically, if $\dim Y \leq k$, and Z is r-connected, then δ_Z is $\rho := \min(2n{-}k{-}2, 2r{+}n)$-connected. Replacing Z be $\Sigma^j Z$ and using the duality map $\Sigma^j d\colon S^{n+j} \to Y \wedge_G \Sigma^j Z$, shows similarly that the corresponding map

$$\delta^j_Z\colon F(Y, S^{n+j}_G)^G \to F(S^{n+j}, S^{n+j} \wedge \Sigma^j Z)$$

is $(\rho{+}2j)$-connected.

On the other hand, there is an evident map

$$\iota^j_Z\colon \Sigma^j Z \to F(S^{n+j}, S^{n+j} \wedge \Sigma^j Z)\,.$$

If $\dim Z \le \ell$, it follows that $\dim \Sigma^j Z \le \ell + j$, and eventually, $\rho + 2j$ exceeds $\ell + j$ as j-increases. By equivariant obstruction theory we obtain a morphism $q\colon \Sigma^j Z \to F(Y, S_G^{n+j})^G$ such that $\delta_Z^j \circ q$ is equivariantly homotopic to ι_Z^j. Furthermore, q is unique up to equivariant homotopy. We obtain therefore a composite map

$$\Sigma^j Z \xrightarrow{q} F(Y, S_G^{n+j})^G \xrightarrow{\delta_{Z'}^j} F(S^{n+j}, S^{n+j} \wedge \Sigma^j Z') .$$

If s denotes the connectvity of Z' then the map

$$\iota_{Z'}^j \colon \Sigma^j Z' \to F(S^{n+j}, S^{n+j} \wedge \Sigma^j Z')$$

is $(2s+2j+1)$-connected. By requiring j to be large, we can guarantee the inequality $\ell + j < 2s + 2j + 1$. Again by equivariant obstruction theory, we see that there exists a morphism $h\colon \Sigma^j Z \to \Sigma^j Z'$ such that $\iota_{Z'}^j \circ h$ coincides with $\delta_{Z'}^j \circ q$ up to equivariant homotopy, and h is unique up to equivariant homotopy. It is automatic that h is a weak equivalence, and that $(\mathrm{id}_Y \wedge_G h) \circ d$ is homotopic to d'. □

7. Second proof of 4.1

Let K be a connected space. Choose a basepoint for K. Let $S.K$ be the simplicial total singular complex. Let $G.$ denote the Kan loop group of $S.K$, and set $G = |G.|$ (with compactly generated topology). Then there is a functorial identification $BG \simeq K$, where BG is the classifying space of G (i.e., the bar construction of $G.$). Using this identification we can assume $K = BG$ without any loss in generality.

If we set $\tilde{K} := EG$, then have a fibration $\tilde{K} \to K$. For any map of spaces $A \to K$, let $\tilde{A} \to \tilde{K}$ be the associated pullback along $\tilde{K} \to K$. Then $\tilde{A} \subset A \times \tilde{K}$ inherits an (unbased) G-action (here G acts on the product by acting trivially on the first coordinate and by its usual action on EG on the second coordinate). In the homotopy category of spaces over K, the Borel construction $\tilde{K} \times_G \tilde{A} \to K$ is canonically isomorphic to $A \to K$

For technical reasons, we replace $\tilde{A}$ by its cofibrant approximation $(\tilde{A})^{\mathrm{c}} \xrightarrow{\sim} \tilde{A}$ in $R^G(*)$. By abuse of notation, let $\tilde{K}/\tilde{A}$ be the mapping cone of the composite $(\tilde{A})^{\mathrm{c}} \xrightarrow{\sim} \tilde{A} \to \tilde{K}$. Then $\tilde{K}/\tilde{A}$ is a cofibrant object of $R^G(*)$. Then the reduced diagonal map $\tilde{K}/\tilde{A} \to \tilde{K}_+ \wedge \tilde{K}/\tilde{A}$ is equivariant and therefore induces a map of G-orbits

$$\Delta\colon (\tilde{K}/\tilde{A})_G \to \tilde{K}_+ \wedge_G \tilde{K}/\tilde{A}$$

which is easily seen to be a weak equivalence of based spaces. Up to homotopy, $(\tilde{K}/\tilde{A})_G$ is just $\bar{K}/A$, the mapping cone of $A \to K$.

Second proof of 4.1. (Existence). Let K be a connected homotopy finite space. Then $\tilde{K}_+$ is a homotopy finite object of $R^G(*)$. By the existence part of 6.5, there exist an integer $n \geq 0$, a homotopy finite object Z and an equivariant duality map

$$d\colon S^n \to \tilde{K}_+ \wedge_G Z\,.$$

Let $A = \tilde{K} \times_G Z$ be the Borel construction. The we have a map $A \to K$, and the mapping cone of the latter is identified with $\tilde{K}_+ \wedge_G \Sigma Z$ up to homotopy. If α denotes the map $\Sigma d\colon S^{n+1} \to \tilde{K}_+ \wedge_G \Sigma Z \simeq \bar{K}/A$, then (A, α) is a Spivak $(n{+}1)$-thickening of K.

(Uniqueness). If (A, α) and (A, α') denote a Spivak thickenings of K, by decompressing a sufficient number of times, we can assume that they are both Spivak n-thickenings.

Poincaré duality for $(\bar{K}, A)$ implies that the the composite

$$d\colon S^n \xrightarrow{\alpha} \bar{K}/A \simeq \tilde{K}/\tilde{A} \xrightarrow{\Delta} \tilde{K}_+ \wedge_G \tilde{K}/\tilde{A}$$

satisfies equivariant duality with respect to $h(\mathbb{Z}[\pi_1(K)])$. Application of 6.4 shows d to be an equivariant duality map. Similarly, so is the composite

$$d'\colon S^n \xrightarrow{\alpha'} \bar{K}/A' \simeq \tilde{K}/\tilde{A}' \xrightarrow{\Delta} \tilde{K}_+ \wedge_G \tilde{K}/\tilde{A}'\,.$$

By the uniqueness part of 6.5, there exist an integer $j \geq 0$ and an equivariant weak equivalence $h\colon \Sigma^j \tilde{K}/\tilde{A} \xrightarrow{\sim} \Sigma^j \tilde{K}/\tilde{A}'$ such that $(\mathrm{id}_{\tilde{K}_+} \wedge_G h) \circ \Sigma^j d$ is homotopic to d'.

The Borel construction $\tilde{K} \times_G \Sigma^j \tilde{K}/\tilde{A} \to K$ coincides with the $(j{+}1)$-fold fiberwise suspension $\Sigma_K^{j+1} A \to K$ in the homotopy category of spaces over K. Consequently, h induces an isomorphism in the homotopy category of spaces over K between $\Sigma_K^{j+1} A \to K$ and $\Sigma_K^{j+1} A' \to K$. Moreover, the fact that $(\mathrm{id}_{\tilde{K}_+} \wedge_G h) \circ \Sigma^j d$ is homotopic to d' implies that this isomorphism transfers $\Sigma^{j+1}\alpha$ to $\Sigma^{j+1}\alpha'$. Hence (A, α) and (A, α') are stably concordant. □

8. Proof of the Addendum to Theorem A

Let (ν_P, α) and (ν_M, β) represent the Spivak fibrations for P and M, where say, $\alpha\colon S^{n+k} \to P^{\nu_P}/(\partial P)^{i^*\nu_P}$ and $\beta\colon S^{n+k} \to M^{\nu_M}/(\partial M)^{j^*\nu_M}$ (here Let $i\colon \partial P \to P$ and $j\colon \partial M \to M$ denote the inclusions). Let $\phi\colon \nu_P \simeq f^*\nu_M$ be a fiber homotopy equivalence.

As in the proof of Theorem A(1) (i.e., 3.1(1)), an immersion of f induces a stable collapse map

$$\beta'\colon S^{n+k} \xrightarrow{\beta} M^{\nu_M}/(\partial M)^{j^*\nu_M} \to P^{f^*\nu_M}/(\partial P)^{(f\circ i)^*\nu_M}$$

The fiber homotopy equivalence $\phi' \colon \nu_P \simeq f^*\nu_M$ provided in Theorem A(1) is given by applying the uniqueness result for the Spivak fibration with respect to the pairs $(f^*\nu_M, \beta')$ and (ν_P, α). Using ϕ, we obtain an identification of $(f^*\nu_M, \beta')$ with (ν_P, α'), where α' is obtained composing the identification $P^{f^*\nu_M}/(\partial P)^{(f\circ i)^*\nu_M} \simeq P^{\nu_P}/(\partial P)^{i^*\nu_P}$ (defined by ϕ) with β'.

It it were true that α and α' coincide up to homotopy, then the uniqueness of the Spivak fibration would imply that ϕ and ϕ' coincide up to fiber homotopy. Hence, it will suffice to modify the immersion in such a way that α' coincides with α up to homotopy.

Let $\mathrm{Aut}(\nu_P)$ denote the group of (stable) fiber homotopy automorphisms of ν_P, and let $\pi_{n+k}(P^{\nu_P}/(\partial P)^{i^*\nu_P})^\natural \subset \pi_{n+k}(P^{\nu_P}/(\partial P)^{i^*\nu_P})$ denote the subset of those elements γ satisfying the condition that $U \cap \gamma_*([S^{n+k}])$ is a fundamental class. Then, by [Wa2, 3.5], there is an isomorphism of sets

$$\mathrm{Aut}(\nu_P) \xrightarrow{\cong} \pi_{n+k}(P^{\nu_P}/(\partial P)^{i^*\nu_P})^\natural$$

defined by the assignment $\psi \mapsto \psi' \circ \alpha$, where ψ' is the self-map of $P^{\nu_P}/(\partial P)^{i^*\nu_P}$ which ψ induces. It follows that there exists a unique ψ up to fiber homotopy such that $\alpha' \simeq \psi' \circ \alpha$.

Let τ represent an inverse for ν_P in Sph_P. Then the fiber homotopy automorphism $h := \psi + \mathrm{id}_\tau$ of the trivial fibration induces an automorphism of the object $\partial(P \times D^j) \cong \Sigma_P^{\nu_P+\tau}\partial P$ in the homotopy category of spaces over P. The self-map of $\Sigma^j P^{\nu_P}/(\partial P)^{i^*\nu_P}$ induced by Thomifying h coincides with $\Sigma^j \psi'$ up to homotopy.

Let

$$\begin{array}{ccc} \partial(P\times D^j) & \longrightarrow & W \\ \downarrow & & \downarrow \\ P\times D^j & \xrightarrow[f\times \mathrm{id}_{D^j}]{} & X\times D^j \end{array}$$

be the embedding diagram for the given immersion. Change the map $\partial(P\times D^j) \to W$ by prefixing it with h. The resulting diagram will only homotopy commute, but this can be repaired by replacing W by a suitable mapping cylinder. The result of this procedure yields a new immersion whose stable collapse map coincides with α up to homotopy.

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