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0. Introduction. This paper constitutes a summary of the author's Ph.D. thesis [K]. Proofs of the results cited here will appear elswhere. The first section is devoted to outlining a means of passing in a continuous way from the space of pairs (M, f), where M is a compact smooth manifold and f is a Morse function on M, into a moduli space for finite cell complexes.

In section two the results of section one are applied in special instances to construct a new invariant which is a parametrized analogue of Reidemeister torsion. This invariant takes values in a certain subquotient of higher algebraic K-groups of the complex numbers.

1. Manifold bundles and families of cell complexes. Suppose $p:E \longrightarrow B$ is a bundle over a finite CW complex having compact smooth manifold fibres. Consider a continuous function f on E whose restriction to each fibre of p is a smooth function with no degenerate critical points. Then f is a family of Morse functions parametrized by points of B.[1] In the unparametrized case, that is, when B is a point, a classical procedure [M] shows how to construct from f a finite cell complex Y having the homotopy type of E, in which the number of cells of Y is equal to the number of critical points of f.

However, the usual method of assigning the cell complex Y to the function f is ambiguous unless extra data is chosen (e.g. a Riemannian metric, local coordinates, deformation retractions etc.). It would therefore be natural to ask whether or not the set of cell complexes associated with a given Morse function forms a contractible space. Unfortunately, this is *not* true for the standard construction [M].

By a different, coordinate free approach we prove,

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[1] It is not always true that such a function always exists on a given bundle $p:E \longrightarrow B$. A necessary, but not sufficient obstruction when B is simply connected is that there should be a section of p.

Theorem A. If N is a smooth manfold and $f:N \longrightarrow \mathbb{R}$ is a Morse function, then there is a contractible space C(f) which is intrinsically defined in terms of N and f and which has the property that each point of C(f) uniquely determines a cell complex Y arising from $f:N \longrightarrow \mathbb{R}$.

Now consider the case when B is any finite CW complex. We address the question of whether it is possible or not to naturally associate to a manifold bundle $p:E \longrightarrow B$ together with a fibre-wise Morse function $f:E \longrightarrow \mathbb{R}$, a *bundle of cell complexes* parametrized by points of B. By this, we mean a fibration $\pi:Y \longrightarrow B$ such that for each $b \in B$ the fibre $Y_b = \pi^{-1}(b)$ has the structure of a finite cell complex in such a way that the attaching maps for the cells of Y_b vary continuously with respect to $b \in B$. By the theory of classifying spaces, it turns out that it is sufficient to answer this question in the *universal* case, i.e., for the bundle,

$$p_u:(EG\times\mathfrak{M}(N))\times_G N \longrightarrow B^{\mathfrak{M}}G,$$

where N is the fibre over the basepoint of p, $G \subseteq \mathrm{Diff}(N)$ is a group of diffeomorphisms, EG is a free, contractable G-space, $\mathfrak{M}(N)$ is the space of Morse functions on N (with the Whitney C^∞ topology), and $B^{\mathfrak{M}}G$ denotes the Borel construction $EG\times_G\mathfrak{M}(N)$. Note that this bundle is equipped with a universal fibre-wise Morse function f_u defined by $f_u((x,f),n) \doteq f(n)$. Consequently, the space $B^{\mathfrak{M}}G$ may be viewed as a classifying space for bundles with Morse functions. A positive answer is then provided by the following:

Theorem B. There is a space B^CG and a forgetful map $F:B^CG \longrightarrow B^{\mathfrak{M}}G$ such that

(i) F is a weak homotopy equivalence, and

(ii) each point $b \in B^CG$ with $F(b) = (x,f) \in EG\times_G\mathfrak{M}(N) = B^{\mathfrak{M}}G$ determines a cell complex Y_b associated to the Morse function $f:N \longrightarrow \mathbb{R}$, and furthermore, Y_b varies continuously with respect to $b \in B^CG$.

The importance of theorem B is spelled out in the following corollary:

Corollary. If $f:E \longrightarrow \mathbb{R}$ is a fibre-wise Morse function on a manifold bundle $p:E \longrightarrow B$, then it is always possible to perturb f by a fibre-wise homotopy to yield a fibre-wise Morse function g on $p:E \longrightarrow B$ having the property that there exists a parametrized family of cell complexes $q:Y \longrightarrow B$ which is associated to g.

We sketch a rough outline of the proof of theorem B. The proof involves constructing a G-space $C(N)$, together with a natural G-equivariant map $h:C(N) \longrightarrow \mathfrak{M}(N)$ which is also a weak equivalence, such that the points of $C(N)$ determine cell complexes. We may then set B^cG equal to the Borel contruction $EG\times_G C(N)$. One might *a priori* guess that $C(N)$ is constructed in a sheaf-like manner from the space of Morse functions $\mathfrak{M}(N)$ by defining the stalk of h over a Morse function f to be the space $C(f)$ of theorem A. However, the assignment $f \longmapsto C(f)$ unfortunately has the property that the cell complexes in $C(f)$ do not necessarily vary continuously with respect to the parameter $f \in \mathfrak{M}(N)$. The reason for the discontinuity is that cell complexes in $C(f)$ arise in part by choosing a set of regular values $r_1,\ldots,r_k$ of f that separate the collection of critical values (these define level surfaces in N which separate the critical points). If f_t is a family of Morse functions, it is possible that the number of distinct critical values of f_t is different for different t. Hence the critical values may pass through each other. Consequently, it might not be possible to choose a continuously varying set of regular values $r_1(t),\ldots,r_k(t)$ that separate the critical values of f_t.

To resolve the discontinuity problem, Igusa's *stratification theorem* [I_2; chapter III] is applied to the space $\mathfrak{M}(N)$. Let $\psi:\mathfrak{M}(N) \longrightarrow \mathbb{N}$ be the function which assigns to a Morse function f the difference between its number of critical points and its number of critical values. Then ψ defines a stratification of $\mathfrak{M}(N)$ by setting $\mathfrak{M}(N)_{(i)} = \psi^{-1}(i)$. It is not difficult to see that the assignment $f \longmapsto C(f)$ varies continuously if we remain inside a single stratum, for, within a connected component of a stratum, the relative arrangement of the critical values of functions is fixed. In essence, the idea then is to modify the definition of $C(f)$ so that the cell complexes vary continuously within the closure of a stratum in $\mathfrak{M}(N)$, and to then apply the stratification theorem to glue all of the strata together.

For a function f in the closure of a particular stratum, the aforesaid modification of $C(f)$ is obtained by generalizing the concept of level surface. We choose a collection of oriented hypersurfaces H_α having the following properties:

(i) The H_α are transverse to the 1-form df,

(ii) The H_α partition the critical points into subsets so that if f is in the stratum itself, then the H_α separate critical points having different critical heights.

(iii) If f is in the boundary of a stratum $\mathfrak{S}$, we require that the hypersurfaces H_α are chosen in such a way that a perturbation f_t of f into the interior of $\mathfrak{S}$ can be extended to a perturbation $H_\alpha(t)$ of hypersurfaces satisfying the condition that $H_\alpha(t)$ separates the critical heights of f_t.

2. Higher Reidemeister torsions. My interest in associating families of cell complexes to bundles with fibre-wise Morse function arises from the problem of defining a parametrized analogue of the classical R-torsion invariant of Franz, Reidemeister, and de Rham. This question was first raised by Wagoner [Wa].[2]

Let $\rho:\pi \longrightarrow U_r(\mathbb{C})$ be a unitary representation of the fundamental group of a manifold M. We assume that the homology of M in the local system defined by ρ is entirely vanishing; we then say that M is *acyclic* with respect to ρ (cf. [Wa]). It is in this context that the classical R-torsion is defined, and it lives in a certain quotient of the first algebraic K-group of $\mathbb{C}$.

Let k_ρ be the kernel of ρ and set $\pi_\rho = \pi/k_\rho$. Let $M(\pi_\rho)$ denote the infinite monomial matrices with coefficients in π_ρ. Then $M(\pi_\rho)$ is a subgroup of the stabilized general linear group $GL(\mathbb{C}) = \lim GL_n(\mathbb{C})$. Taking classifying spaces and then plus constructions, we get a map $BM(\pi_\rho)^+ \longrightarrow BGL(\mathbb{C})^+$. Let $Wh^\rho{}_{i+1}(\mathbb{C})$ denote the i^{th}-homotopy group of the homotopy fibre this map. For $i = 0$, $Wh^\rho{}_{i+1}(\mathbb{C})$ is precisely the group in which the classical R-torsion lives.

Now suppose that a smooth manifold bundle $p:E \longrightarrow B$ is given whose fibres are ρ-acyclic. We assume the structure group of p has the property that its action on the fibres of p are basepoint preserving and furthermore has the property that the induced action on fundamental groups is trivial. Suppose that a fibre-wise Morse function f on p is given. Choose a Riemannian structure on the fibres of p. If $b \in B$, let $f_b:E_b \longrightarrow \mathbb{R}$ denote the

[2] Wagoner also provides a solution to this problem in the 1-parameter case.

restriction of f to the fibre of p over b. By a *framing* ϕ_b for f at a point $b \varepsilon B$, we mean an orthonormal framing for the negative eigenspace of $D^2 f_b$ along the singularities of f_b. We shall say that (f,ϕ) is a *framed fibre-wise Morse function* on $p{:}E \longrightarrow B$ if a framing ϕ_b for f_b at every point $b \varepsilon B$ is given which varies continuously with respect to the parameter space B.[3]

Consider the case of a ρ-acyclic bundle over the n-sphere: $p{:}E \longrightarrow S^n$. Suppose a framed fibre-wise Morse function (f,ϕ) is given on p.

Theorem C. *(Higher R-torsion for framed fibre-wise Morse functions).* The triple (p,f,ϕ) determines a well defined element $\tau^\rho(p,f,\phi) \varepsilon \mathrm{Wh}^\rho{}_{n+1}(\mathbb{C})$.

The proof of theorem C is deduced from theorem B together with a construction called *linearization* which allows one to pass from an n-parameter family of cell complexes to an element of the group $\mathrm{Wh}^\rho{}_{n+1}(\mathbb{C})$. In the unparametrized case $n = 0$, this construction coincides with the usual construction of the R-torsion from the cellular chain complex associated to the Morse function f (assuming that f is self indexing).

3. Reidemeister torsions for all ρ-acyclic bundles. A still unsettled question is whether it is possible to define invariants τ^ρ for *all* ρ-acyclic manifold bundles over the n-sphere. I will briefly mention how one might accomplish this. Let $\mathfrak{M}^{fr}(N)$ denote the space of framed Morse functions on N with the Whitney C^∞ topology. Then $\mathfrak{M}^{fr}(N)$ is a G-space where G is the group of diffeomorphisms of N which preserve the base-point and which induce the identity on fundamental groups. The proof of theorem C above follows from the construction of a homomorphism $\tau^\rho{:}\pi_n(B^{\mathfrak{M}^{fr}}G) \longrightarrow \mathrm{Wh}^\rho{}_{n+1}(\mathbb{C})$. Consequently, higher R-torsions can be defined for all bundles over spheres if τ^ρ can be extended to $\pi_n(BG)$. Let $\mathfrak{L}(N)$ be the space of *framed functions* on N, i.e., functions having only Morse and birth–death singularities together with framings of their critical points ([I_1]). Igusa's *framed function theorem* says that $\mathfrak{L}(N)$ has connectivity equal to $\dim(N) - 1$. Using the G-action on $\mathfrak{L}(N)$ defined by precomposition, form the Borel construction $B^{\mathfrak{L}}G = EG \times_G \mathfrak{L}(N)$. Then $B^{\mathfrak{M}^{fr}}G$ is a subspace of $B^{\mathfrak{L}}G$, and the forgetful map $B^{\mathfrak{L}}G \longrightarrow BG$ has

[3] In terms of cell complexes, a framed function defines an explicit Euclidean coordinatization of the cells.

connectivity $\geq \dim(N)$. Suppose that τ^ρ can be extended to $\pi_n(B^{\mathfrak{Q}}G)$. If the dimension of N is made large by replacing N with $N \times D^k$ for $k >> n$ *(stabilization)*, then the forgetful map $B^{\mathfrak{Q}}G \longrightarrow BG$ becomes highly connected and therefore τ^ρ is extendable to $\pi_n(BG)$ for all n. In order to construct an extension of τ^ρ to $\pi_n(B^{\mathfrak{Q}}G)$, it would be sufficient to show how a bundle $p:E \longrightarrow B$ with fibre-wise framed function (f,ϕ) gives rise to a family of cell complexes $q:Y \longrightarrow B$, where it is now necessary to include Whitehead *elementary collapses* and *elementary expansions* in the transition between the fibres of $p:E \longrightarrow B$, as is reflected by the existence of birth-death singularities in the fibre-wise restrictions of f. In a future paper with K. Igusa [I-K], this program will be undertaken.[4]

4. A remark on the machinery employed. The proofs in [K] apply the language of Waldhausen's theory of *categories with cofibrations and weak equivalences* [W], which enlarges the class of categories for which one can define K-theory. We use a particular model of Waldhausen's of algebraic K-theory of a space called *the expansion space*, which is a type of moduli space for cell complexes [I-W]. The primary issue in the proof of theorem C is to pass continuously from the moduli space of Manifold, framed Morse function pairs (i.e. $B^{\mathfrak{M}^{fr}}G$) into the expansion space. We then show how to linearize from the expansion space into the algebraic K-theory of the complex numbers. The higher R-torsion is, by definition, the composition of these two constructions.

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[4] added remark: K. Igusa and I have recently shown by other methods how define higher torsion invariants for all ρ-acyclic manifold bundles.