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The dualizing spectrum of a topological group

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Abstract. To a topological group G, we assign a naive G-spectrum D_G, called the *dualizing spectrum of* G. When the classifying space BG is finitely dominated, we show that D_G detects Poincaré duality in the sense that BG is a Poincaré duality space if and only if D_G is a homotopy finite spectrum. Secondly, we show that the dualizing spectrum behaves multiplicatively on certain topological group extensions. In proving these results we introduce a new tool: a *norm map* which is defined for any G and for any naive G-spectrum E. Applications of the dualizing spectrum come in two flavors: (i) applications in the theory of Poincaré duality spaces, and (ii) applications in the theory of group cohomology. On the Poincaré duality space side, we derive a homotopy theoretic solution to a problem posed by Wall which says that in a fibration sequence of finitely dominated spaces, the total space satisfies Poincaré duality if and only if the base and fiber do. The dualizing spectrum can also be used to give an entirely homotopy theoretic construction of the Spivak fibration of a finitely dominated Poincaré duality space. We also include a new proof of Browder's theorem that every finite H-space satisfies Poincaré duality. In connection with group cohomology, we show how to define a variant of Farrell-Tate cohomology for any topological or discrete group G, with coefficients in any naive equivariant cohomology theory E. When E is connective, and when G admits a subgroup H of finite index such that BH is finitely dominated, we show that this cohomology coincides with the ordinary cohomology of G with coefficients in E in degrees greater than the cohomological dimension of H. In an appendix, we identify the homotopy type of D_G for certain kinds of groups. The class includes all compact Lie groups, torsion free arithmetic groups and Bieri-Eckmann duality groups.

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1. Introduction

In this paper the symbol G will denote either the realization of a simplicial group or a Lie group. Let

$$S^0[G]$$

(the "group ring of G over the sphere spectrum") denote the suspension spectrum of G_+, i.e., the spectrum whose j-th space is $Q(S^j \wedge (G_+))$, where Q is the stable homotopy functor (here and elsewhere, G_+ denotes the union of G with a disjoint basepoint).

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Let $G\times G$ act on G by the rule $(g,h)*x = gxh^{-1}$. This induces a left action of $G\times G$ on $S^0[G]$.

Definition. The *dualizing spectrum* of G is the homotopy fixed point spectrum of the subgroup $G = G\times 1 \subset G\times G$ acting on $S^0[G]$:

$$D_G \ = \ S^0[G]^{hG} := F(EG_+, S^0[G])^{G\times 1}\,.$$

This is a G-spectrum, whose action is given by restriction to the subgroup $1\times G \subset G\times G$.

We emphasize that this work will only employ the *naive* kind of equivariant spectra (that is, the group doesn't act on the suspension coordinates).

Motivation. In making this definition, we were prompted by a similar construction arising in the theory of group cohomology. Given a discrete group Γ whose classifying space is finitely dominated, one considers

$$\mathcal{D}_\Gamma := \hom_{D(\mathbb{Z}[\Gamma])}(\mathbb{Z}, \mathbb{Z}[\Gamma])$$

where hom is taken internally within the *derived category* of (left) $\mathbb{Z}[\Gamma]$-modules (the homology of this complex is, of course, $\mathrm{Ext}^*_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, \mathbb{Z}[\Gamma])$). One calls $\mathcal{D}_\Gamma$ a *dualizing module* if it is isomorphic in the derived category to a complex which is non-trivial in a single degree $-n$ and which in that degree is torsion free as an abelian group (compare [Br4, Ch. VIII Th. 10.1]). If $\mathcal{D}_\Gamma$ is a dualizing module, then $B\Gamma$ satisfies a version of Poincaré duality in which the fundamental class lives in $H_n(BG; \mathcal{D}_\Gamma)$. Such groups Γ are called *Bieri-Eckmann duality groups.*

By analogy, our dualizing spectrum is given by replacing the discrete Γ by the (possibly) continuous G, and the integers $\mathbb{Z}$ by the sphere S^0:

$$D_G = \hom_{D(S^0[G])}(S^0, S^0[G])\,,$$

where hom is now taken internally in the derived category of naive G-spectra.

An important property of the dualizing spectrum is its ability to detect Poincaré duality in the classifying space BG:

Theorem A. *Assume that BG is finitely dominated. Then the following are equivalent:*

1. *BG is a Poincaré duality space,*
2. *D_G has the (unequivariant) weak homotopy type of a sphere,*
3. *D_G is unequivariantly homotopy finite.*

Furthermore, in (1) *BG has formal dimension n, if and only if in* (2) *D_G is a sphere of dimension $-n$, if and only if in* (3) *D_G has non-trivial spectrum homology in degree $-n$.*

The implication 1 $\Leftrightarrow$ 2 is also due to Bill Dwyer (independently). Theorem A will be proved in §5. In §10 (cf. Sect. 10 Ex. 1 and 10.5) we refine Theorem A by identifying the *equivariant* weak homotopy type of D_G for those groups having finitely dominated classifying space.

Remark 1.1. Any connected based space X can be regarded up to homotopy as BG for a suitable topological group G (take G to be a topological group model for the based loop space of X). Consequently, Theorem A characterizes the class of Poincaré duality spaces.

Furthermore, if one regards the Borel construction

$$EG \times_G D_G \to BG$$

as a "family of spectra" parametrized by points of BG, then Theorem A shows that this fibration stably spherical precisely when BG is a Poincaré space. We show in Corollary 5.1 that the above is just the Spivak normal fibration of BG. We therefore have a purely homotopy theoretic construction the Spivak normal fibration.

Another feature of the dualizing spectrum is that it behaves *multiplicatively* with respect to certain kinds of extensions. Suppose that

$$1 \to H \to G \to Q \to 1$$

is an extension.

Theorem B. *Assume either that*

- *the classifying spaces BH, BG and BQ are finitely dominated, or that*
- *BH is a finitely dominated Poincaré space.*

Then there is a weak equivalence of spectra

$$D_G \simeq D_H \wedge D_Q .$$

Actually, Theorem B can be made equivariant. Call a map of G-spectra an *equivariant weak equivalence* if it induces an isomorphism on homotopy groups. More generally, two G-spectra X and Y are equivariantly weak equivalent, written $X \simeq_G Y$, if there exists a finite zig-zag of such morphisms starting with X and ending at Y.

Using the fact that H is normal in G, it is possible to replace D_H by a G-spectrum D'_H up to canonical weak equivalence of H-spectra (cf. 2.6). Also, since Q acts on D_Q as well, G acts on D_Q by restriction using the homomorphism $G \to Q$. Thus, we may give $D'_H \wedge D_Q$ the associated diagonal G-action.

Addendum C. *With respect to the hypotheses of Theorem B, there is a weak equivalence of G-spectra $D_G \simeq_G D'_H \wedge D_Q$.*

The norm map. An important tool of this paper is the existence of a norm map relating 'invariants = group cohomology' to 'coinvariants = group homology.'

Theorem D. *For any G-spectrum E, there is a (weak) map*

$$D_G \wedge_{hG} E \to E^{hG}$$

(natural in E) called the "norm map" which is a weak equivalence if one of the following holds:

- *G is arbitrary and E is G-finitely dominated in the sense that it is a retract up to homotopy of a spectrum built up from a point by attaching a finite number of free G-cells, or*
- *BG is finitely dominated and E is arbitrary, or*
- *G is a compact Lie group and E is an induced spectrum (in the sense that E has the equivariant weak homotopy type of a spectrum of the form $W \wedge G_+$).*

Conversely, assume that $\pi_0(G)$ is finitely presented, and that the norm map is a weak equivalence for all G-spectra. Then BG is finitely dominated.

Remarks 1.2. (1). The domain of the norm map is the homotopy orbit spectrum of G acting diagonally on $D_G \wedge E$. The codomain of the norm map is the homotopy fixed point spectrum of G acting on E.

(2). To see how the norm map connects with Poincaré duality, consider the case when $E = HM$ is the Eilenberg-Mac Lane spectrum on a $\pi_0(G)$-module M. If BG is finitely dominated, and if D_G is unequivariantly a sphere of fixed dimension $-n$, say, then applying homotopy groups to the norm map gives an isomorphism

$$H_{n-*}(G; \mathcal{D}_G \otimes M) \cong H^*(G; M)\,,$$

where $\mathcal{D}_G$ denotes $\pi_{-n}(D_G)$. But this means that BG is a Poincaré duality space. (This gives the $2 \Rightarrow 1$ implication of Theorem A.)

(3). Assume BG is finitely dominated. Then the fact that the norm map is a weak equivalence shows that taking homotopy fixed points with respect to G commutes with homotopy colimits of G-spectra.

(4). The proof of Theorem D appears in Sect. 3, except in the instance when G is a compact Lie group and E is induced. The proof of the latter appears separately in 10.2. It is a consequence of the identification in the compact Lie case that

$$S^0[G] \;\simeq_{G\times G}\; F(G_+, S^{\mathrm{Ad}_G})\,,$$

where the right side is a function spectrum of maps from G_+ to $S^{\mathrm{Ad}_G} =$ the suspension spectrum of the one point compactification of the adjoint representation of G.

It should be true, although I haven't verified it, that our norm map in the compact Lie case coincides with the norm map of Adem, Cohen and Dwyer [A–C–D] and Greenlees and May [G–M].

The homotopy type of D_G. In the appendix, we identify the weak homotopy type of the dualizing spectrum for various kinds of groups (sometimes it will be possible to identify the weak *equivariant* homotopy type). Here is the list of such groups (in the following, Γ always refers to a discrete group, while G can be either discrete or continuous; a † indicates that the weak equivariant homotopy type is identified):

1. BG is a connected finitely dominated Poincaré duality space†
2. G is a compact Lie group†
3. G is a finitely dominated topological group
4. Γ is a discrete cocompact subgroup of a connected Lie group
5. $\Gamma = \mathbb{Z}^{*g}$ is the free group on g generators†
6. $\Gamma = P_n$ is the pure braid group on n-strings
7. $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2$ is the infinite dihedral group
8. Γ is a torsion free arithmetic group†
9. Γ is a Bieri-Eckmann duality group
10. $\Gamma = \mathbb{Z}^d * \mathbb{Z}^m$

An application to group cohomology. The existence of the norm map enables one to define a version of Farrell-Tate cohomology for an arbitrary group and an arbitrary G-spectrum:

Definition 1.3. *Let E^{tG} be the homotopy cofiber of the norm map*

$$D_G \wedge_{hG} E \to E^{hG} .$$

Define the generalized Farrell-Tate cohomology of G with coefficients in E to be the homotopy groups of E^{tG}:

$$\widehat{E}^*(G) := \pi_{-*}(E^{tG}) .$$

Remark 1.4. If G is discrete and and M is a G-module then we recover the case of Tate cohomology by taking $E = HM$ (the Eilenberg-Mac Lane spectrum of M). More generally, Farrell-Tate cohomology theory is defined when G is discrete and has finite virtual cohomological dimension (cf. [Br4, Ch. X]).

In another paper [Kl1] we will show that the homotopy groups of the cofiber of the norm map coincide with the Farrell-Tate groups in two instances: (a) When G is finite (i.e., the Tate case), or (b) when M admits a finite type projective resolution over $\mathbb{Z}[G]$.

By the Theorem D, $\widehat{E}^*(G) = 0$ if BG is finitely dominated. What happens if G has a subgroup H of finite index such that BH is finitely dominated? The following result shows that in high degrees one recovers the group cohomology of G with coefficients in the spectrum E. Define $E^*(G)$ to be $\pi_{-*}(E^{hG})$. Then

Theorem E. *Assume that E is connective (i.e., (-1)-connected). Suppose that G admits a subgroup H of finite index such that*

- *BH is finitely dominated.*
- *$H^*(BH; M) = 0$ for $* > n$ and any local coefficient bundle M on BH.*

Then there is an isomorphism

$$\widehat{E}^*(G) \cong E^*(G) \qquad \text{if } * > n\,.$$

If G is discrete, then in the language of group cohomology, the hypotheses of the theorem amount to saying that G is VFP and has virtual cohomological dimension $\le n$. In the classical situation when $E = H\mathbb{Z}$ and G is discrete, the theorem specializes to one of the well-known properties of Farrell-Tate cohomology.

Applications to Poincaré duality spaces. Suppose that $F \to E \to B$ is a fibration of connected, finitely dominated spaces. Choose a basepoint for F. Applying a suitable group model for the loop space, we obtain an extension of topological groups

$$1 \to \Omega F \to \Omega E \to \Omega B \to 1\,.$$

(Details: let $\Omega_. E$ and $\Omega_. B$ denote the Kan loop groups of the total singular complex of E and B. Define $\Omega_. F$ to be the kernel of the onto homomorphism $\Omega_. E \to \Omega_. B$. Then apply realization.)

Since the smash product of two spectra is weak equivalent to the sphere spectrum if and only if each constituent is,[1] it follows that $D_{\Omega E}$ is a sphere if and only if $D_{\Omega F}$ and $D_{\Omega B}$ are spheres. Applying Theorem B, we have

Corollary F. *With respect to the above assumptions, E is a Poincaré space if and only if F and B are.*

The corollary has a history. C.T.C. Wall first posed the statement as a question, and a solution was announced by Quinn (unpublished, but see [Qu2]). A proof involving manifold techniques was first published by Gottlieb [Go]. The present proof is homotopy theoretic.

Suppose that $X \subseteq \mathbb{R}^n$ is the compact regular neighborhood of a connected finite polyhedron. Assume that the spine of X has codimension ≥ 3 (this can be

[1] *Proof:* Suppose that $X \wedge Y \simeq S^0$, and that X and Y are CW Ω-spectra. By the Künneth formula, it is sufficient to show that X and Y are homotopy finite spectra. Since we have a weak equivalence of hom-spaces $\hom(X, -) \simeq \hom(S^0, Y \wedge -)$, we may infer that $\hom(X, -)$ commutes with colimits. If we write X as a colimit of finite CW spectra, it follows that the identity map of X factors up to homotopy through some finite spectrum. We infer that X is homotopy finite. Similarly, so is Y. I wish to thank T. Goodwillie and S. Schwede for showing me this argument.

arranged, if necessary, by embedding X in a higher dimensional euclidean space). Let F denote the homotopy fiber of the inclusion of the boundary $\partial X \to X$. Since ∂X is a closed manifold, we have

Corollary G. *The space X satisfies Poincaré duality if and only if F is homotopy finite.*

The ‘if’ part follows directly from Corollary F, whereas the ‘only if’ part is well-known. In fact, F has the homotopy type of a sphere in this instance. Furthermore, this is the procedure that is usually employed to construct the Spivak fibration (see e.g., [Br1, I.4.1] which relies on [Br1, I.4.3]; for another kind of proof of the latter, see [Kl2]).

Another application of the dualizing spectrum is a new proof of an historically important theorem of W. Browder [Br2] concerning finite H-spaces (where ‘H-space’ now means ‘Hopf space’ = space with multiplication up to homotopy):

Theorem H. *A connected finitely dominated H-space satisfies Poincaré duality.*

Note: Browder asserted this only for finite H-spaces and for Poincaré duality with $\mathbb{Z}$-coefficients, so we are actually asserting more. The idea of the new proof runs as follows: if X is a finitely dominated H-space, then we shall prove that the dualizing spectrum of (a topological group model for) its loop space ΩX is unequivariantly homotopy finite. Then the claim that X is a Poincaré duality space follows from Theorem A.

Observe the similarity of Addendum C with what happens in the smooth case: if $p\colon E \to B$ is a smooth submersion of compact manifolds, then we have the splitting $\tau_E = \tau_E^{\text{fib}} \oplus p^*\tau_B$ where τ_E^{fib} is the tangent bundle along the fibers. Equivalently, in terms of stable normal bundles $\nu_E = \nu_E^{\text{fib}} \oplus p^*\nu_B$. This last fact is the analogue of our addendum if we regard D_G together with its G-action as an object akin to the stable normal bundle. This connection will be made precise in 5.1 and in Sect. 10. In the case of Poincaré spaces, we will prove:

Theorem I. *Let $F \to E \to B$ be a fibration of connected finitely dominated spaces. If E is a Poincaré space then its Spivak fibration ν_E has the stable fiber homotopy type of a fiberwise join*

$$\nu_F^{\text{fib}} *_B p^*\nu_B \, ,$$

where ν_F^{fib} is a certain prolongation of the Spivak fibration of F to a spherical fibration over E and $p^*\nu_B$ denotes the pullback of the Spivak fibration of B to E.

Yet another consequence of our machinery is a result which says fibrations of connected finitely dominated spaces admit fiberwise Poincaré space thickenings:

Theorem J (Fiber Poincaré Thickening). *Let $F \to E \to B$ be a fibration of connected finitely dominated spaces. Then there is a fibration pair*

$$(F', \partial F') \to (E', E'_0) \to (B, B)$$

such that $F' \to E' \to B$ is fiber homotopy equivalent to $F \to E \to B$ and $(F', \partial F')$ is a Poincaré space.

A note on methods. This paper relies heavily on the paper [Kl5]. The proofs of the results listed above are homotopy theoretic. There is only one place in the paper where a manifold argument appears: in the appendix, in order to identify the dualizing spectrum of a compact Lie group, we use the exponential map (cf. 10.1). This result is then used to show that the norm map is a weak equivalence for induced spectra (cf. 10.2).

Outline. Section 2 is primarily language and basic homotopy invariant constructions which can be applied to equivariant spectra. Section 3 is about the proof of Theorem D. In Sect. 4, we prove Theorem B and Addendum C. Section 5 contains the proof of Theorem A. In Sect. 6 we prove Theorem H. The proof of Theorem I is in Sect. 7. In Sect. 8 we prove Theorem J, and in §9 we prove Theorem E. Section 10 is the appendix, in which we identify the dualizing spectrum of different kinds of groups, and end the discussion with a problem, a question and a conjecture.

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I originally thought of the dualizing spectrum as a gadget assigned to a topological space (i.e., $X \mapsto D_{\Omega X}$). I am grateful to Greg Arone, who suggested that my norm map should be related to the classical norm map in the compact Lie case. Although his suggestion is not verified in this paper, it did lead to thinking of the dualizing spectrum as a gadget assigned to a topological group. The latter point of view turned out to be ultimately the more fruitful one.

2. Preliminaries

Spaces. All spaces below will be compactly generated, and **Top** will denote the category of compactly generated spaces. In particular, we make the convention that products are to be retopologized with respect to the compactly generated topology. Let $\mathbf{Top}_*$ denote the category of compactly generated based spaces. A *weak equivalence* of spaces is shorthand for (a chain of) weak homotopy equivalence(s). A weak equivalence is denoted by $\xrightarrow{\sim}$, whereas, we often write chains of weak equivalences using $\simeq$ (the same notation will be used when discussing weak equivalences of spectra). A space is *homotopy finite* it is weak equivalent to a finite CW complex. It is *finitely dominated* if it is a retract up to homotopy of a finite CW complex.

Homotopy colimits of diagrams of spaces are formed by applying the total singularization functor, taking the homotopy colimit of the resulting diagram of simplicial sets (as in [B–K]) and thereafter applying the realization functor.

If X is a connected based space, we associate a topological group object G of **Top** as follows: let $S_{.}X$ denote the simplicial total singular complex of X, and let $G_{.}$ denote its Kan loop group. Define G to be the geometric realization of the underlying simplicial set of $G_{.}$. The assignment $X \mapsto G$ is a functor. Moreover, there is a functorial chain of weak homotopy equivalences connecting BG to X.

Remark 2.1. In this paper a "topological group" always means either: (i) the realization of a simplicial group, or (ii) a Lie group.

Poincaré spaces. A space X is a *Poincaré duality space* of (formal) dimension n if there exists a bundle of coefficients $\mathcal{L}$ which is locally isomorphic to $\mathbb{Z}$, and a fundamental class $[X] \in H_n(X;\mathcal{L})$ such that the associated cap product homomorphism

$$\cap[X]\colon H^*(X;M) \to H_{n-*}(X;\mathcal{L}\otimes M)$$

is an isomorphism in all degrees. Here, M denotes any bundle of coefficients.

Usually, Poincaré spaces are implicitly understood to have some sort of finiteness condition imposed upon them. For the most part, we shall assume that X is a finitely dominated CW complex.

More generally, a CW pair $(X,\partial X)$ is a *Poincaré pair* of dimension n if there exists a bundle of coefficients $\mathcal{L}$ which is locally isomorphic to $\mathbb{Z}$, and a fundamental class $[X] \in H_n(X,\partial X;\mathcal{L})$ such that the associated cap product homomorphism

$$\cap[X]\colon H^*(X;M) \to H_{n-*}(X,\partial X;\mathcal{L}\otimes M)$$

is an isomorphism in all degrees for all local coefficient bundles M, and furthermore, $\partial_*[X] \in H_{n-1}(\partial X;\mathcal{L}_{|\partial X})$ equips ∂X with the structure of a Poincaré duality space. See [Wa2] or [Wa3] for more details.

Spectra. A *spectrum* will be taken to mean a collection of based spaces $\{X_i\}_{i\in\mathbb{N}}$ together with based maps $\Sigma X_i \to X_{i+1}$ where ΣX_i denotes the reduced suspension of X_i. A *map of spectra* $X \to Y$ consists of maps $X_i \to Y_i$ which are compatible with the structure maps.

Let G be a topological group. A *(naive) G-spectrum* consists of a spectrum X such that each X_i is a based (left) G-space and each structure map $\Sigma X_i \to X_{i+1}$ is equivariant, where the action of G on ΣX_i is defined so as to act trivially on the suspension coordinate.

Homotopy groups are defined in the usual way. Maps of G-spectra are maps of spectra that are compatible with the G-action. Let $\mathbf{Sp}^G$ denote the category of these. One way to obtain a G-spectrum is to take a based G-space X and form its *suspension spectrum* $\Sigma^\infty X$; the j-th space of the latter is $Q(S^j \wedge X)$, where $Q = \Omega^\infty \Sigma^\infty$ is the stable homotopy functor. In particular, $S^0[G]$ is the suspension spectrum of G_+.

A *weak equivalence* of G-spectra is a morphism inducing an isomorphism on homotopy groups. Weak equivalences are indicated by $\xrightarrow{\sim_G}$, and we say that two G-spectra X and Y are *weak equivalent*, written $X \simeq_G Y$, if there is a finite chain of weak equivalences, starting at X and terminating at Y.

A map of spectra is *r-connected* if it induces a surjection on homotopy up through degree r and an isomorphism in degrees less than r. A spectrum is r-connected if the map to the trivial spectrum (consisting of the one point space in each degree) is $(r+1)$-connected. A spectrum is *bounded below* if it is r-connected for some r.

S. Schwede has shown that the above notion of weak equivalence arises from a Quillen model category structure on $\mathbf{Sp}^G$ (cf. [Sc]). In this model structure, a *fibrant* object is a G-spectrum X which is an Ω-spectrum: the adjoint $X_n \to \Omega X_{n+1}$ to the structure maps are weak homotopy equivalences. A *cofibrant* object is (the retract of) a G-spectrum X such that X_n is built up from a point by attaching *free* G-cells (i.e., $D^n \times G$), moreover, the structure maps $\Sigma X_n \to X_{n+1}$ are given by attaching free G-cells to ΣX_n.

Any G-spectrum X has a (functorial) *cofibrant approximation:* there exists a cofibrant G-spectrum X^{c} and a weak equivalence $X^{\mathrm{c}} \xrightarrow{\sim} X$ (in fact X^{c} can be constructed by the usual procedure of killing homotopy groups). Similarly, X has a (functorial) *fibrant approximation:* there exists a fibrant G-spectrum X^{f} and a weak equivalence $X \xrightarrow{\sim} X^{\mathrm{f}}$ (this can be constructed by taking X^{f}_n to be the homotopy colimit $\operatorname{hocolim}_j \Omega^j X_{n+j}$.)

Generally, we will assume that the collection of spaces describing a G-spectrum are CW complexes. If the result Y of a construction on X fails to have this property, we apply the functor $Y_n \mapsto |S.Y_n|$, the realization of the singularization functor. The result gives a G-spectrum which is degreewise a CW complex.

Smash products and functions with spaces. If U is a G-space and X is a G-spectrum, then $U \wedge X$ will denote the G-spectrum which in degree j is the smash product $U \wedge X_j$ provided with the diagonal action. This has the correct homotopy type if the underlying space of U is a CW complex. (Here and elsewhere, we say that a construction gives the "correct homotopy type" if it respects weak equivalences. Thus, the functor $U \mapsto U \wedge X$ respects weak equivalences whose domain and codomain are CW complexes.)

Give $U \wedge X$ the diagonal G-action. Then we can from the *orbit spectrum*

$$U \wedge_G X$$

given by taking G-orbits degreewise. In general, the latter has the correct homotopy type if U is a based G-CW complex which is free away from the basepoint.

Similarly, we can form the function spectrum $F(U, X)$ which in degree j is given by $F(U, X_j) =$ the function space of unequivariant based maps from U to X_j. An action of G on $F(U, X)$ provided by conjugation (i.e., $(g * f)(u) = gf(g^{-1}u)$ for $g \in G$ and $f \in F(U, X_j)$). In general, for $F(U, X)$ to have the correct homotopy type, it is necessary to assume that X is fibrant and that the underlying space of U is a CW complex.

Let

$$F(U, X)^G$$

denote the *fixed point spectrum* of G acting on $F(U, X)$, i.e., the spectrum whose j-th space consists of the equivariant functions from U to X_j. The fixed point spectrum has the correct homotopy type if X is fibrant and U is a based G-CW complex which is free away from the base point.

In what follows below we sometimes abuse notation: if X fails to be fibrant (but U is a based G-CW complex which is free away from the basepoint), we take $F(U, X)^G$ to mean $F(U, X^{\mathrm{f}})^G$.

Smash products of equivariant spectra. We will *not* require internal smash products of spectra which are *strictly* associative, commutative and unital. However, we will require that these have been defined so as to be *homotopy* associative, commutative and unital.

In particular, a naive type construction will suffice for our purposes: if X is a G-spectrum and if Y is an H-spectrum then $X \wedge Y$ is the $(G \times H)$-spectrum whose $(2n)$-th space is $X_n \wedge Y_n$ and whose $(2n+1)$-st space is $X_{n+1} \wedge Y_n$. If $H = G$, then G acts diagonally on $X \wedge Y$. We can then form the associated *orbit spectrum* $X \wedge_G Y$. This has the correct homotopy type provided that X or Y is cofibrant.

Suppose X, Y and Z are spectra, and that we are given maps $f_{ij} : X_i \wedge Y_j \to Z_{i+j}$ compatible with the structure maps of X, Y and Z. Then we obtain a map of spectra $X \wedge Y \to Z$.

Homotopy orbits and homotopy fixed points. If X is a G-spectrum then the *homotopy orbit spectrum* X^{hG} is the (non-equivariant) spectrum given by

$$X \wedge_G EG_+ ,$$

where EG is the free contractible G-space (arising from the bar construction), and EG_+ is the result of adding a basepoint to EG.

The *homotopy fixed point spectrum* X^{hG} is given by

$$F(EG_+, X)^G .$$

(recall that our conventions specify $F(EG_+, X)^G$ to mean $F(EG_+, X^{\mathrm{f}})^G$ whenever X fails to be fibrant).

Lemma 2.2. *Let X be a bounded below G-spectrum. For any $\mathbb{Z}[\pi_0(G)]$-module M, let HM denote the corresponding Eilenberg-Mac Lane spectrum, with G acting by means of the homomorphism $G \to \pi_0(G)$.*

Suppose that the homotopy orbit spectrum

$$X \wedge_{hG} HM$$

is r-connected for every M. Then X is r-connected.

Proof. First assume that G is connected. It is shown in [Kl2, Lemma 1.3] that if G is connected, X is bounded below and X_{hG} is weakly contractible, then X is also weakly contractible. In proving this we actually showed the stronger statement that if X_{hG} is r-connected, then X is r-connected. The Hurewicz theorem (for bounded below spectra) shows that X_{hG} is r-connected if $X \wedge H\mathbb{Z}$ is r-connected. This gives the result when G is connected.

When G isn't connected, we can reduce to the connected situation as follows: notice that $X \wedge_{hG} HM$ coincides up to homotopy with $X_{hG_0} \wedge_{h\pi_0(G)} HM$, where G_0 is the kernel of $G \to \pi_0(G)$. Take M to be $\mathbb{Z}[\pi_0(G)]$. It follows that $(H\mathbb{Z} \wedge X)_{hG_0}$ is r-connected. But G_0 is connected. Therefore $H\mathbb{Z} \wedge X$ is also r-connected by the previous paragraph. The Hurewicz theorem now enables one to conclude that X is r-connected. □

Homotopy invariance of D_G. Suppose that $H_\cdot \to G_\cdot$ is a monomorphism of simplicial groups. Taking realization we get a closed monomorphism $H \to G$ of topological groups. Then D_G is also an H-spectrum by restriction.

Lemma 2.3. *With respect to the above hypotheses, assume in addition that $H \to G$ induces an isomorphism on homotopy groups. Then there is an equivariant weak equivalence*

$$D_H \ \simeq_H \ D_G .$$

Proof. Note that EG also serves as a model for EH. The equivariant weak equivalence of dualizing spectra is given by the chain

$$F(EG_+, S^0[G])^G \xrightarrow{\sim_H} F(EG_+, S^0[G])^H \xleftarrow{\sim_H} F(EG_+, S^0[H])^H .$$

where the first map is the inclusion of G-fixed sets into H-fixed sets and the second map is induced by the inclusion $S^0[H] \to S^0[G]$. □

Induced spectra. Let $H \to G$ is a homomorphism, and let X be an H-spectrum. Then one may form the *induced spectrum*, the G-spectrum given by

$$X \wedge_H G_+ \,,$$

where the action in degree j is defined by

$$g * (x, \gamma) \quad := \quad (x, \gamma g^{-1}) \,,$$

with $g \in G, \gamma \in G_+, x \in X_j$.

Now assume that $H \subset G$ is the closed inclusion of a normal subgroup. Let $Q = G/H$. If X happens to be a G-spectrum to begin with, then the induced spectrum $X \wedge_H G_+$ comes equipped with $(G \times Q)$-action: the action of Q is defined by

$$k * (x, \gamma) \quad := \quad (\hat{k}x, \hat{k}\gamma)$$

for $k \in Q, x \in X_j, \gamma \in G_+$ and $\hat{k} \in G$ denoting any representative lift of k.

For $g \in G$, let $\bar{g} \in Q$ denote its image. Let $g \in G$ act on Q_+ by the rule $g * x = x(\bar{g})^{-1}$. If Z is a G-spectrum, give $Z \wedge Q_+$ the associated diagonal action.

The following is probably well-known.

Lemma 2.4. *Assume that X is a G-spectrum. Then there is a weak equivalence*

$$X \wedge_H G_+ \;\simeq_{G \times Q}\; X \wedge Q_+ \,.$$

In particular, taking $H = G$, there is a weak equivalence

$$X \wedge_G G_+ \;\simeq_G\; X \,.$$

Proof. If Y is a G-space then there is an homeomorphism of G-spaces

$$Y \wedge_H G_+ \;\cong\; Y \wedge Q_+$$

defined by $(y, g) \mapsto (g^{-1}y, \bar{g})$. This map of spaces extends to the spectrum level to define the equivalence. □

Coinduced spectra. If $H \subset G$ is a closed subgroup, and E is a (fibrant) H-spectrum, then we can form the G-spectrum

$$F(G_+, E)^H$$

This is the effect of *coinducing* E with respect to the inclusion $H \to G$. G acts on $F(G_+, E)^H$ by $(g * \phi)(x) = \phi(g^{-1}x)$, where $g \in G$ and $\phi \colon G_+ \to E_j$.

If G/H is discrete, then $F(G_+, E)^H$ can be rewritten as the cartesian product

$$\prod_{G/H} E \,.$$

Similarly, the induced spectrum $E \wedge_H G_+$ may be rewritten as a wedge

$$\bigvee_{G/H} E \, .$$

Now if H has finite index in G, it follows that the inclusion of the wedge into the product is a weak equivalence. Consequently, there is an equivariant weak equivalence

$$E \wedge_H G_+ \simeq_G F(G_+, E)^H$$

provided that H has finite index in G (compare with the 'linear analogue' [Br4, Prop. 5.9]).

Lemma 2.5. *Suppose that $H \subset G$ has finite index. Then there is an unequivariant weak equivalence*

$$D_H \ \simeq \ D_G \, .$$

Proof. We have

$$D_G = F(EG_+, S^0[G])^G \ \simeq \ F(EG_+, S^0[H] \wedge_H G_+)^G \, .$$

Replacing the induced spectrum $S^0[H] \wedge_H G_+$ by the coinduced spectrum $F(G_+, S^0[H])^H$ we obtain

$$D_G \ \simeq \ F(EG_+, F(G_+, S^0[H])^H)^G \, .$$

Taking the adjunction, we have that

$$F(EG_+, F(G_+, S^0[H])^H)^G = F(EG_+ \wedge G_+, S^0[H])^{H \times G} \, .$$

Note that H acts only on the G_+ factor of $EG_+ \wedge G_+$, whereas G acts diagonally. The second factor projection $EG_+ \wedge G_+ \to G_+$ is therefore a $(G \times H)$-equivariant weak equivalence. But G_+ isn't $(G \times H)$-free; we can make it $(G \times H)$-free at the expense of smashing with EH_+ (with the trivial G-action and the usual H-action). This entitles us to replace $EG_+ \wedge G_+$ with $EH_+ \wedge G_+$ in the function spectrum. Consequently,

$$D_G \simeq F(EH_+ \wedge G_+, S^0[H])^{H \times G} = F(G_+, F(EH_+, S^0[H])^H)^G = D_H \, .$$

$\square$

Extending the action. Let

$$1 \to H \to G \to Q \to 1$$

denote an extension.

Consider the restriction map

$$D_H := F(EH_+, S^0[H])^H \ \overset{\sim_H}{\leftarrow} \ F(EG_+, S^0[H])^H := D'_H$$

which is induced by inclusion $EH \to EG$. Since EG_+ is a G-space, we can let G act on D'_H by means of the formula

$$g * \phi \quad = \quad (s \mapsto g\phi(g^{-1}s)g^{-1}),$$

where $g \in G$, $\phi \in F(EG_+, Q(S^j \wedge H_+))^H$ and $s \in EG_+$.

This requires some explanation: G acts on $Q(S^j \wedge H_+)$ by conjugation on H (this makes sense, since H is normal in G). Moreover, notice that if $g \in H$, then the H-equivariance of ϕ gives

$$g * \phi \quad = \quad (s \mapsto \phi(s)g^{-1}).$$

Consequently, the G-action we have defined on D'_H actually extends the naturally given H-action. Summarizing, we have

Lemma 2.6. *The map $D'_H \to D_H$ of H-spectra is a weak equivalence. Moreover, the H-action on D'_H extends to a G-action in a canonical way.*

3. The norm map

In this section we prove Theorem D, except in the case when G is a compact Lie group and E is induced. That case is handled separately in 10.2 below.

Construction of the norm map. The task is to construct a weak map

$$D_G \wedge_{hG} E \to E^{hG}$$

which is natural in E. By applying fibrant and cofibrant replacement to E, we can assume without loss in generality that E is fibrant and cofibrant. Then it suffices to define a map

$$D_G \wedge_G E \to E^{hG}$$

(where the domain now has orbits instead of homotopy orbits).

Recall once again that $S^0[G]$ has a $(G\times G)$-action, i.e., a pair of commuting G-actions. In order to differentiate between them, we let G_ℓ denote the subgroup $G\times 1$ and G_r the subgroup $1\times G$. Thus $D_G := F(EG_+, S^0[G])^{G_\ell}$ is a G_r-spectrum. Similarly, we let $*_r$ denote the G_r action and $*_\ell$ the G_ℓ-action.

For integers $j, k \geq 0$, define a map

$$N_{j,k}\colon F(EG_+, S^0[G]_j)^{G_\ell} \wedge_{G_r} E_k \to (S^0[G]_j \wedge_{G_r} E_k)^{hG_\ell}$$

by the rule

$$(x, e) \;\mapsto\; (v \mapsto (x(v), e))$$

for $x \in F(EG_+, S^0[G]_j)^{G_\ell}$ and $e \in E_k$. This is well defined: if $g \in G$, then we have $(x, e) \sim (g *_r x, ge)$, where $(g *_r x)(v) = x(v)g^{-1}$. But $N_{j,k}(g *_r x, ge)$ is the function $v \mapsto (x(v)g^{-1}, ge) \sim x(v)$. Hence, $N_{j,k}(x,e) = N_{j,k}(g *_r x, ge)$. Therefore, $N_{j,k}$ is invariant under the G_r-action.

We still need to check that $N_{j,k}$ maps into the homotopy fixed set. That is, we must show that function $v \mapsto (x(v), e)$ is G_ℓ-equivariant. If $g \in G_\ell$, then we calculate

$$\begin{aligned} N_{j,k}(x,e)(gv) := (x(gv), e) = (gx(v), e) &=: g *_\ell (x(v), e) \\ &= g *_\ell (N_{j,k}(x,e)(v)) . \end{aligned}$$

Consequently, we land in the homotopy fixed set.

The map $N_{j,k}$ just constructed is compatible with the indices as j and k vary. Hence, we obtain a map of spectra

$$N : D_G \wedge_G E \to (S^0[G] \wedge_{G_r} E)^{hG} .$$

On the other hand, there is a natural identification of G-spectra

$$E \ \simeq_G \ S^0[G] \wedge_{G_r} E ,$$

so we may consider N as a weak map

$$D_G \wedge_{hG} E \to E^{hG} .$$

This completes the construction of the norm map.

Remark 3.1. There is a more straightforward way to think of the construction, provided one is willing to admit that the homotopy category of G-spectra has *internal* function objects. The norm map may then be defined as the *composition pairing*

$$\hom(S^0, S^0[G]) \ \wedge_{S^0[G]} \ \hom(S^0[G], E) \to \hom(S^0, E) ,$$

where hom is taken in the homotopy category of G-spectra.

We are now ready to establish the properties of the norm map.

The case when G is arbitrary and E is G-finitely dominated. By 2.4, the norm map is clearly a weak equivalence when $E = S^0[G]$ (since the target in this case is precisely D_G and the norm map is identified with the identity in this case). By (de-)suspending, the norm map is a weak equivalence for the spectrum $S^k \wedge G_+$, where $k \in \mathbb{Z}$ is any integer.

Suppose that $E = E' \cup (D^{k+1} \wedge G_+)$ is the result of attaching a cell to a G-spectrum E', and suppose that the norm map is a weak equivalence for E'. We have a homotopy cofiber sequence of G-spectra

$$S^k \wedge G_+ \to E' \to E .$$

Since cofiber sequences are up to homotopy fiber sequences, it follows that we have an associated homotopy cofiber sequence

$$(S^k \wedge G_+)^{hG} \to (E')^{hG} \to E^{hG}$$

and the five lemma shows that the norm map is a weak equivalence for E. Hence the norm map is a weak equivalence for any G-homotopy finite spectrum. Naturality, and the fact that retracts preserve weak equivalences then shows that we get a weak equivalence for any G-finitely dominated spectrum.

The case when BG is finitely dominated and E is arbitrary. The procedure of killing homotopy groups shows that E can be expressed up to homotopy as a filtered homotopy colimit of G-spectra E^α, where α is an index and E^α is a G-spectrum having a finite number of (free) cells – in particular, the norm map is a weak equivalence for E^α. Since BG is finitely dominated, it follows that EG_+ is a G-finitely dominated based G-space (in the sense that up to equivariant homotopy, it is a *retract* of a based G-space built up from a point by attaching a finite number of (free) cells). The 'small object argument' now applies, yielding a weak equivalence of spectra

$$\operatorname*{hocolim}_\alpha F(EG_+, E^\alpha)^G \ \simeq\ F(EG_+, E)^G \,.$$

From this equivalence, it is straightforward to deduce that the norm map is a weak equivalence for E.

The partial converse. Suppose that the norm map is a weak equivalence for all spectra E, and that $\pi_0(G)$ is finitely presented. The task is now to show that BG is finitely dominated. The idea of the proof is that since $D_G \wedge_{hG} E \simeq E^{hG}$, it follows that homotopy fixed points commutes with arbitrary homotopy colimits (since homotopy orbits does).

To proceed, we substitute for E the sphere spectrum S^0 with trivial G-action, and choose a weak equivalence

$$\operatorname*{hocolim}_\alpha E^\alpha \ \overset{\sim_G}{\to}\ S^0$$

in which E^α is a finite G-spectrum (this is accomplished by the procedure of killing homotopy groups).

Because taking homotopy fixed sets commutes with filtered homotopy colimits, the associated map

$$\operatorname*{hocolim}_\alpha (E^\alpha)^{hG} \to (S^0)^{hG}$$

is a weak equivalence.

Consider the equivariant map $c\colon EG_+ \to S^0$ which is given by collapsing EG to the non-basepoint of S^0. Because the displayed map is a weak equivalence, there exist an index α and an equivariant homotopy factorization of (the stable map associated with) c:

$$S^0 \wedge EG_+ \to E^\alpha \to S^0 .$$

Let G_0 denote the identity component of G. Taking homotopy orbits with respect to G_0, we obtain a $\pi_0(G)$-equivariant homotopy factorization

$$S^0 \wedge \tilde{B}G_+ \to E^\alpha_{hG_0} \to S^0 \wedge \tilde{B}G_+ ,$$

where $\tilde{B}G = EG/G_0$ is the universal cover of BG (note: $\tilde{B}G$ is a model for BG_0). Take the smash product with the Eilenberg-Mac Lane spectrum $H\mathbb{Z}$ and identify the resulting Eilenberg-Mac Lane $\pi_0(G)$-spectra with $\mathbb{Z}[\pi_0(G)]$-chain complexes. It follows that the singular chain complex of $\tilde{B}G_+$ is dominated by the homotopy finite chain complex corresponding to the Eilenberg-Mac Lane spectrum (with $\pi_0(G)$-action) $H\mathbb{Z} \wedge E^\alpha_{hG_0}$. By a result of Wall [Wa2] it follows that BG is finitely dominated. This completes the proof of Theorem D. □

Remark 3.2. In proving the partial converse, note that we actually proved more: to show that BG is finitely dominated, we only need to assume that $\pi_0(G)$ is finitely presented and that the norm map is a weak equivalence for $E = S^0$.

To apply Theorem D in the proof of Theorem B and Addendum C, we will require an equivariant version of the norm map. Suppose that

$$1 \to H \to G \to Q \to 1$$

is an extension. Let E be an $(L \times G)$-spectrum, where L is yet another topological group.

Consider the space $F(EG_+, E_j)$. This again admits a $(L \times G)$-action given by

$$(\ell, g) * \phi \quad := \quad (s \mapsto (\ell, g)\phi(g^{-1}s)) .$$

for $g \in G$ and $\ell \in L$, and $\phi \colon EG_+ \to E_j$. Taking fixed points with respect to H identified as the subgroup $1 \times H \subset L \times G$, and letting j vary, it follows that the spectrum

$$F(EG_+, E)^H$$

comes equipped with an $(L \times Q)$-action.

Corollary 3.3. *Assume that E is an $(L \times G)$-spectrum. Then there is an $(L \times Q)$-equivariant (weak) map*

$$D'_H \wedge_{hH} E \to F(EG_+, E)^H ,$$

where D'_H is the G-spectrum of 2.6. Furthermore, this map coincides up to homotopy with the norm map for E considered as an H-spectrum.

Proof. We first explain how each of the spectra in the statement of 3.3 are equivariant. We have already indicated how $(L\times Q)$ acts on $F(EG_+, E)^H$. Give $D'_H \wedge_{hH} E$ an L-action using the given L-action on E and the trivial L-action on D'_H. Since $D'_H \wedge E$ may be given the diagonal G-action, the homotopy orbit spectrum $D'_H \wedge_{hH} E$ has a Q-action. Consequently, $D'_H \wedge_{hH} E$ has a $(L\times Q)$-action.

We next explain how the map is defined (the reader may wish at this point to consult the construction of the norm map as given in the proof of Theorem D). Assume without loss in generality that E is fibrant and cofibrant.

For indices $j, k \geq 0$ there is a map of spaces

$$\tilde{N}_{j,k}\colon F(EG_+, S^0[H]_j)^H \wedge_H E_k \to F(EG_+, S^0[H]_j \wedge_H E_k)^H$$

given by $(x, e) \mapsto (v \mapsto (x(v), e)$ (this is the same formula we used to define the norm map). By a straightforward check which we omit, $\tilde{N}_{j,k}$ is well-defined. We claim that $\tilde{N}_{j,k}$ is $(L\times Q)$-equivariant. L equivariance is clear (L behaves like a dummy variable). Let $\bar{g} \in Q$ be any element, and let $g \in G$ denote any lift of it. Equivariance with respect to Q follows from the calculation

$$\begin{aligned} \tilde{N}_{j,k}(\bar{g} * (x, e))(v) &= (\tilde{N}_{j,k})((g * x, ge))(v) = (gx(g^{-1}v), ge) \\ &=: (\bar{g} * \tilde{N}_{j,k}(x, e))(v)\,. \end{aligned}$$

Letting the indices j, k now vary, we obtain a (weak) map of $(L\times Q)$-spectra

$$D'_H \wedge_{hH} E \to F(EG_+, E)^H$$

Unequivariantly, this (clearly) is identified with the norm map. □

4. Proof of Theorem B and Addendum C

Let

$$1 \to H \to G \to Q \to 1$$

be an extension

Since $D_G = S^0[G]^{hG}$, and $Q = G/H$, we have a weak equivalence of G-spectra

$$D_G \simeq_G (F(EG_+, S^0[G])^H)^{hQ}\,. \tag{1}$$

We first consider the inside term $F(EG_+, S^0[G])^H$. Since $S^0[G]$ has a $(G\times G)$-action, taking homotopy fixed points with respect to H identified as the subgroup $1\times H \subset G\times G$, and applying Corollary 3.3 together with Theorem D, we obtain a weak equivalence of $(G\times Q)$-spectra

$$F(EG_+, S^0[G])^H \simeq_{G\times Q} D'_H \wedge_{hH} S^0[G] \simeq_{G\times Q} D'_H \wedge_H G_+\,. \tag{2}$$

Here we are using the fact that BH is finitely dominated.

By 2.4 we also have a weak equivalence

$$D'_H \wedge_H G_+ \simeq_{G\times Q} D'_H \wedge Q_+ \,. \tag{3}$$

Assembling, we get a weak equivalence

$$F(EG_+, S^0[G])^H \simeq_{G\times Q} D'_H \wedge Q_+ \,. \tag{4}$$

Take homotopy fixed points of both sides of this with respect to Q (considered as the subgroup $1\times Q \subset G\times Q$). Since we are assuming either: (i) BQ is finitely dominated, or (ii) that BH is a finitely dominated Poincaré space so that D'_H is a sphere,[2] and therefore $D'_H \wedge Q_+$ is Q-finitely dominated, we are in a position to apply Theorem D and Corollary 3.3 again to obtain weak equivalences of G-spectra

$$\begin{aligned} D_G &\simeq_G (F(EG_+, S^0[G])^H)^{hQ} && \text{by (1)} \\ &\simeq_G (D'_H \wedge Q_+)^{hQ} && \text{by (4)} \\ &\simeq_G D_Q \wedge_Q (D'_H \wedge Q_+) && \text{by Theorem D and 3.3} \\ &\simeq_G D_Q \wedge D'_H && \text{by 2.4.} \end{aligned}$$

This completes the proof of Theorem B and Addendum C. □

5. Proof of Theorem A

Assume that BG is finitely dominated.

‘$2 \Rightarrow 3$’: Trivial.

‘$3 \Rightarrow 1$’: This will use Theorem B and the unstable equivariant duality theory developed in [Kl5, §6] (see also [Kl3]). To explain this will require some preparation.

Recall that if X and Y are based G-CW complexes which are free away from the basepoint, then an *equivariant duality* is a map $d\colon S^n \to X \wedge_G Y$ such that the associated map of function spectra

$$F(X, E)^G \xrightarrow{\sim} F(S^n, E \wedge_G Y) \,.$$

is a weak equivalence for any G-spectrum E (the correspondence is given by $f \mapsto (f \wedge_G \mathrm{id}_Y) \circ d$).

It is shown in [Kl5, Th. 6.5] that for any G-homotopy finite based free G-CW complex X, there exist an integer $n \gg 0$, a G-homotopy finite free based G-CW complex Y and an equivariant duality $S^n \to X \wedge_G Y$. By a straightforward

[2] See ‘$1 \Rightarrow 2$’ in the proof of Theorem A appearing in the next section. Note that we aren’t arguing circularly since ‘$1 \Rightarrow 2$’ does not use Theorem B in its proof.

argument which we omit, if X is G-finitely dominated, then there is a G-finitely dominated Y, an $n \gg 0$ and an equivariant duality $S^n \to X \wedge_G Y$. By taking a suitable suspension, we can assume that Y is simply connected.

We are now ready to proceed with the proof. Since BG is finitely dominated (as unbased space) EG_+ is G-finitely dominated. Consequently, there exist an integer $n \gg 0$, a G-finitely dominated Y and an equivariant duality map

$$d\colon S^n \to EG_+ \wedge_G Y\,.$$

Passing to the stable category and n-fold desuspending, we obtain a map of spectra

$$S^0 \to EG_+ \ \wedge_G \ \Sigma^{-n} Y$$

inducing a weak equivalence

$$F(EG_+, E)^G \xrightarrow{\sim} F(S^0, E \wedge_G \Sigma^{-n} Y)$$

for any G-spectrum E. Remember that $S^0[G]$ has a $(G \times G)$-action. We may take E to be $S^0[G]$ with its $(G \times 1)$-action. Therefore we get a weak equivalence

$$D_G := F(EG_+, S^0[G])^G \xrightarrow{\sim} F(S^0, S^0[G] \wedge_G \Sigma^{-n} Y) = \Sigma^{-n} Y\,.$$

By naturality, this weak equivalence is G-equivariant. Thus we conclude that there is an equivariant weak equivalence

$$D_G \simeq_G \Sigma^{-n} Y\,.$$

By assumption, D_G is unequivariantly homotopy finite. From this we infer that Y is an (unequivariant) homotopy finite space.

The pair of Borel constructions

$$(EG \times_G CY, EG \times_G Y)$$

(where CY denotes the cone on Y with G acting trivially on the cone coordinate) is a (finitely dominated) Poincaré pair. Poincare duality is a consequence of two facts: firstly, the quotient associated to the pair is $EG_+ \wedge_G \Sigma Y$, and the statement of equivariant duality for EG_+ with respect to the G-spectrum $E =$ the Eilenberg-Mac Lane spectrum HM on a $\mathbb{Z}[\pi_0(G)]$-module M gives an isomorphism

$$H^*(EG \times_G CY; M) \ \cong \ H_{n+1-*}(EG \times_G CY, EG \times_G Y; M)\,.$$

Secondly, as the inclusion $EG \times_G Y \to EG \times_G CY$ is 2-connected, [Kl4, Lemma 2.1] enables one to conclude that the pair in question is a Poincaré pair.

In particular, the boundary $EG \times_G Y$ is a finitely dominated Poincaré duality space, and the fibration

$$Y \to EG \times_G Y \to BG$$

is a fibration of connected finitely dominated spaces. Consequently, Corollary F shows that BG is a Poincaré duality space.

'1 $\Rightarrow$ 2': If BG is a Poincaré duality space of dimension n say, then it has a Spivak fibration. If we use the method of [Kl5], then the Spivak fibration is given as follows: let

$$S^j \to EG_+ \wedge_G Y$$

be an equivariant duality, where Y is G-finitely dominated and 1-connected. It is shown in the proof of [Kl5, Cor. C] that Y is unequivariantly homotopy equivalent to S^{j-n}, and the Spivak fibration of BG is given by the Borel construction

$$Y \to EG \times_G Y \to BG\,.$$

But we know from arguments above that there is an equivariant weak equivalence $D_G \simeq_G \Sigma^{-j} Y$. Consequently, D_G is unequivariantly weak equivalent to S^{-n}. This finishes the proof of Theorem A. □

In the process of proving Theorem A, observe that we actually established more:

Corollary 5.1. *Suppose that BG is a finitely dominated. Then*

- *D_G is a suspension spectrum, i.e., there is an integer $j \gg 0$ and an equivariant weak equivalence*

  $$\Sigma^j D_G \simeq_G \Sigma^\infty Y$$

  for some G-finitely dominated 1-connected based G-space Y.
- *If furthermore BG is a Poincaré space, then Y is unequivariantly weak equivalent to a sphere and the Spivak fibration of BG is given by the Borel construction*

  $$Y \to EG \times_G Y \to BG\,.$$

We end this section with a corollary which shows that the property of being a Poincaré duality space is preserved with respect to taking finite coverings.

Corollary 5.2. *Suppose $\widetilde{X} \to X$ is a finite covering projection, where $\widetilde{X}$ and X are connected finitely dominated spaces. Then $\widetilde{X}$ is a Poincaré duality space of dimension n if and only if X is.*

Proof. We may assume without loss in generality that $X = BG$. Then $\widetilde{X} \simeq BH$ where $H \subset G$ has finite index and the covering map is given by $BH \to BG$. By 2.5 we have $D_H \simeq D_G$, so D_H is a sphere of dimension $-n$ if and only if D_G is. Now apply Theorem A. □

6. The proof of Theorem H

We first give the proof while ignoring technicalities, and thereafter fill in the details.

Let $G(X)$ be the topological monoid of self homotopy equivalences of X, and let $G(X,*)$ denote the the topological monoid of based equivalences. Then there is a fibration

$$G(X,*) \to G(X) \to X$$

in which the projection from total space to base is given by the evaluation map at the basepoint. Using a suitable group model for the loop space ΩX and $G(X,*)$, the connecting map

$$\Omega X \to G(X,*)$$

is then a homomorphism. It can also be arranged that this map is the inclusion of a normal subgroup (see below). It follows that there is an equivariant weak equivalence

$$D_{\Omega X} \simeq_{\Omega X} D'_{\Omega X}$$

in which the right side has the structure of a $G(X,*)$-spectrum (cf. 2.6). To avoid notational clutter we assume without loss in generality that $D_{\Omega X}$ comes equipped with an extension of its ΩX-action to a $G(X,*)$-action.

Now, the fibration is classified by a map $u\colon X \to BG(X,*)$ which is null homotopic: use the H-space structure on X to get a section up to homotopy $X \to G(X)$ of the evaluation map. Therefore, u factorizes as $X \to CX \to BG(X,*)$. If we loop this factorization back we obtain a factorization of groups

$$\Omega X \to \Omega CX \to G(X,*)\,.$$

where the composite coincides with the connecting map. It follows that ΩX-action on $D_{\Omega X}$ admits an extension to an action of a contractible group.

But this implies that $D_{\Omega X}$ is isomorphic to a spectrum with *trivial* action in the homotopy category of ΩX-spectra: the isomorphism is defined by the chain of weak equivalences

$$D_{\Omega X} \overset{\sim_{\Omega X}}{\leftarrow} D_{\Omega X} \wedge (\Omega CX)_+ \overset{\sim_{\Omega X}}{\to} D^{\mathrm{triv}}_{\Omega X}\,,$$

where $D^{\mathrm{triv}}_{\Omega X}$ means $D_{\Omega X}$ equipped with trivial action and

- the middle term $D_{\Omega X} \wedge (\Omega CX)_+$ is given the diagonal ΩX-action (ΩX acts on $(\Omega CX)_+$ by left translation).
- The left map is defined by projection onto the first factor of the smash product.
- The right map is defined by the formula $(x,t) \mapsto t^{-1}x$, where $x \in (D_{\Omega X})_j$ and $t \in (\Omega CX)_+$.

Step 2. Since $D_{\Omega X}$ is ΩX-finitely dominated, the homotopy orbit spectrum

$$(D_{\Omega X})_{h\Omega X}$$

is homotopy finite.

Since the action of ΩX on $D_{\Omega X}$ is homotopically trivial, it follows that the evident map

$$D_{\Omega X} \to (D_{\Omega X})_{h\Omega X}$$

is a coretraction: homotopical triviality of the action shows that $(D_{\Omega X})_{h\Omega X}$ is identified up to homotopy with

$$D_{\Omega X} \wedge (B\Omega X)_+ \simeq D_{\Omega X} \wedge X_+ ,$$

and a retraction is defined by the map $D_{\Omega X} \wedge X_+ \to D_{\Omega X} \wedge S^0$ that is given by smashing the identity of $D_{\Omega X}$ with the based map $X_+ \to S^0$ given by collapsing X to the non-basepoint of S^0.

Since $D_{\Omega X}$ is a retract of its homotopy orbits, we infer that $D_{\Omega X}$ is homotopy finite when considered as an unequivariant spectrum. By Theorem A, we infer that X is a Poincaré duality space. This completes the outline of the proof.

We now proceed to fill in the details. Instead of looping the classifying map $u\colon X \to BG(X,*)$, we consider instead the map $BG(X,*) \to BG(X)$. Convert this map into a Serre fibration, and call the result $BG(X,*)^{\mathrm{f}} \to BG(X)$. Let us think of u now as a map $X \to BG(X,*)^{\mathrm{f}}$, and choose a null-homotopy $CX \to BG(X,*)^{\mathrm{f}}$.

Let $\omega.$ denote the functor from based spaces to simplicial groups which assigns to a based space its total singular complex followed by its Kan loop group. Let $L.X$ denote the kernel of the homomorphism $\omega.BG(X,*)^{\mathrm{f}} \to \omega.BG(X)$. Then we have a commutative square

$$\begin{array}{ccc} \omega.X & \longrightarrow & L.X \\ \downarrow & & \downarrow \\ \omega.CX & \longrightarrow & \omega.BG(X,*)^{\mathrm{f}} . \end{array}$$

Let $C'_.X$ denote the homotopy pushout in the model category of simplicial groups of the diagram

$$\omega.CX \leftarrow \omega.X \to L.X$$

(see [Qu1]). Then $C'_.X$ is a contractible and the homomorphism $L.X \to \omega.BG(X,*)^{\mathrm{f}}$ factors through $C'_.X$. The realization $LX := |L.X|$ is yet another topological group model for the loop space of X, and the homomorphism $LX \to |\omega.BG(X,*)^{\mathrm{f}}|$ is the inclusion of a normal subgroup.

Consequently, the dualizing spectrum D_{LX} can we modified in its equivariant weak homotopy type to a spectrum D'_{LX} having an action of $|\omega.BG(X,*)^{\mathrm{f}}|$. By commutativity of the above square, the action of LX on D'_{LX} restricts to an action of $\omega X := |\omega.X|$, and the action of the latter extends to the contractible group $\omega C'X := |\omega.C'_.X|$. We conclude from this that the action of ωX on D'_{LX} is homotopically trivial.

Finally, observe that ωX is yet another model for the loop space of X (in particular, $B\omega X$ is homotopy equivalent to X), and that D'_{LX} is ωX-equivariantly weak equivalent to $D_{\omega X}$. So the action of ωX on $D_{\omega X}$ is homotopically trivial. The rest of the proof follows Step (2) above. This completes the discussion of details and the proof of Theorem H. □

7. The proof of Theorem I

The following result will be required for the proof:

Proposition 7.1. *Assume that BG is finite dimensional up to homotopy. Assume that W is a G-spectrum. Suppose that there exists an unequivariant weak equivalence*

$$W \simeq \Sigma^\infty X$$

where X is a finite complex. Then there exist an integer $j \gg 0$, a G-space Z and an equivariant weak equivalence

$$\Sigma^j W \simeq_G \Sigma^\infty Z\,.$$

Proof. By applying fibrant and cofibrant replacement, we can assume without loss in generality that W is fibrant and cofibrant, and that X is a CW complex. Let $\mathrm{Aut}(W)$ denote the topological monoid whose points are (unequivariant) self-maps $W \to W$ which are weak equivalences. The action of G on W specifies a homomorphism of topological monoids $G \to \mathrm{Aut}(W)$, which upon applying classifying spaces, gives a map

$$BG \to B\mathrm{Aut}(W)\,.$$

Let $\Sigma^\infty_{\mathrm{c}} X$ be the spectrum whose j-th space is $S^j \wedge X$. Then $\Sigma^\infty_{\mathrm{c}} X$ is a cofibrant version of the suspension spectrum of X By hypothesis, we may choose an *un*equivariant weak equivalence $\Sigma^\infty_{\mathrm{c}} X \xrightarrow{\sim} W$, where X is a finite dimensional complex. Let $\Sigma^\infty_{\mathrm{c,f}} X$ be the effect of (functorially) converting $\Sigma^\infty_{\mathrm{c}} X$ into a fibrant and cofibrant (unequivariant) spectrum.

We assert that the homomorphism of topological monoids

$$\Phi : \mathrm{Aut}(\Sigma^\infty_{\mathrm{c}} X) \to \mathrm{Aut}(\Sigma^\infty_{\mathrm{c,f}} X)$$

(given by the functor which maps a function to the map induced on fibrant approximations) is a weak equivalence of underlying spaces. To see this, let $\mathcal{E}(\Sigma^\infty_{\mathrm{c}} X, \Sigma^\infty_{\mathrm{c,f}} X)$ be the space of weak equivalences from $\Sigma^\infty_{\mathrm{c}} X$ to $\Sigma^\infty_{\mathrm{c,f}} X$. Then the map

$$\Psi : \mathrm{Aut}(\Sigma^\infty_{\mathrm{c,f}} X) \to \mathcal{E}(\Sigma^\infty_{\mathrm{c}} X, \Sigma^\infty_{\mathrm{c,f}} X)$$

which is given by restricting the source is a weak equivalence of underlying spaces, since both the source and target of Ψ are function spaces having the 'correct' homotopy type (each function space consists of mappings out of a cofibrant object into a fibrant object).

Also the composite

$$\Psi \circ \Phi : \mathrm{Aut}(\Sigma^\infty_{\mathrm{c}} X) \to \mathcal{E}(\Sigma^\infty_{\mathrm{c}} X, \Sigma^\infty_{\mathrm{c,f}} X)$$

is the map given by including targets—it too is a weak equivalence of spaces, since $\Sigma^\infty_{\mathrm{c}}$ has a right adjoint that preserves weak equivalences between cofibrant objects. Consequently, the homomorphism Φ is also weak equivalence of spaces.

Applying classifying spaces, we obtain weak equivalences

$$B\mathrm{Aut}(\Sigma^\infty_{\mathrm{c}} X) \xrightarrow{\sim} B\mathrm{Aut}(\Sigma^\infty_{\mathrm{c,f}} X) \simeq B\mathrm{Aut}(W)$$

where the second of these equivalences arises because the fibrant and cofibrant spectra $\Sigma^\infty_{\mathrm{c,f}} X$ and W are homotopy equivalent.

On the other hand, if $\mathrm{Aut}_*(\Sigma^j X)$ refers to the topological monoid of based self weak equivalences of the space $\Sigma^j X$, the Freudenthal suspension theorem says that the evident homomorphism

$$\mathrm{Aut}_*(\Sigma^j X) \to \mathrm{Aut}(\Sigma^\infty_{\mathrm{c}} X)$$

has connectivity $j - 3 - \dim X$, where $\dim X$ denotes the dimension of X as a CW complex. Assembling, we have maps

$$B\mathrm{Aut}_*(\Sigma^j X) \to B\mathrm{Aut}(W)$$

whose connectivity tends to infinity as j does.

Since BG is homotopy finite dimensional, there exists an integer j such that the map $BG \to B\mathrm{Aut}(W)$ factors up to homotopy through a map

$$BG \to B\mathrm{Aut}_*(\Sigma^j X)\,.$$

This means that we can construct a fibration over BG with fiber $\Sigma^j X$, such that the fibration is equipped with a section. If we pull this fibration back along $EG \to BG$, the resulting total space, call it Y, is an (unbased) space with G-action equipped with an equivariant section $EG \to Y$. Moreover, Y is has the unequivariant homotopy type of $\Sigma^j X$. The mapping cone of this section yields a based G-space Z again having the unequivariant homotopy type of $\Sigma^j X$. A tedious, albeit straightforward, checking of definitions (which we omit) shows that $\Sigma^\infty Z$ and $\Sigma^j W$ are equivariantly weak equivalent. This completes the proof of 7.1. □

Proof of Theorem I. Suppose that

$$F \to E \to B$$

is a fibration of connected finitely dominated Poincaré duality spaces. Choose a basepoint for F (this gives basepoints for E and B). Let G denote the realization of the Kan loop group of the total singular complex of E and let Q be the realization of Kan loop group of the total singular complex of B. Define H to be the kernel of the surjective homomorphism $G \to Q$. Then we have an extension

$$1 \to H \to G \to Q \to 1 .$$

Applying the classifying space functor gives us a fibration $BH \to BG \to BQ$ which is identified with the original fibration up to weak equivalence.

By Addendum C we have an equivariant weak equivalence

$$D_G \simeq_G D'_H \wedge D_Q .$$

By Theorem A, these spectra are all spheres, and since the classifying spaces BH, BG and BQ are finitely dominated, they are also homotopy finite dimensional. Consequently, we may apply 7.1 to conclude that there exist a based Q-space Y, a based G-space Z, an integer $j \gg 0$ and equivariant weak equivalences

$$\Sigma^j D_Q \simeq_Q \Sigma^\infty Y \qquad \text{and} \qquad \Sigma^j D'_H \simeq_G \Sigma^\infty Z .$$

By 5.1, $EQ\times_Q Y \to BQ$ represents the Spivak fibration of BQ. Since the pullback of $EG\times_G Z \to BG$ to BH is identified with $EH\times_H Z \to BH$, and the latter is the Spivak fibration of BH, the former is a prolongation of the latter to BG. Consequently, we have a weak equivalence

$$\Sigma^j D_G \ \simeq_G \ \Sigma^\infty Y \wedge Z ,$$

where G acts diagonally on the right hand side.

By 5.1, the Borel construction $EG\times_G(Y \wedge Z) \to BG$ is the Spivak fibration of BG. It is straightforward to check that this last fibration has the fiberwise stable type of the fiberwise join of the fibrations $EG\times_G Y \to BG$ and $EG\times_G Z \to BG$. □

8. Proof of Theorem J

Let $F \to E \to B$ be a fibration of connected finitely dominated spaces. As in the last section, we can assume that this is coming from an extension of topological groups

$$1 \to H \to G \to Q \to 1$$

by applying the classifying space functor.

Let D'_H be the dualizing spectrum of H modified as in §2 so that it has an extension to a G-action. Then according to 5.1 there exist an integer $j \gg 0$, a 1-connected based G-space Z and an equivariant weak equivalence

$$D'_H \simeq_H \Sigma^\infty Z .$$

Then $(EG\times_H CZ, EG\times_H Z)$ is a Poincaré pair (details omitted; the argument is essentially the same which is used in the proof of '$3 \Rightarrow 1$' in Theorem A), and the fibration pair

$$(EG\times_H CZ, EG\times_H Z) \to (EG\times_G CZ, EG\times_G Z) \to (BQ, BQ)$$

completes the proof. □

9. Proof of Theorem E

Let M be any $\mathbb{Z}[\pi_0(H)]$-module. Since EH_+ is finitely dominated, we know and that D_H is equivariantly dual to EH_+ (cf. the proof of $3 \Rightarrow 1$ in Theorem A). Consequently, there is an isomorphism

$$\pi_{-*}(D_H \wedge_{hH} HM) \cong H^*(BH; M) = 0 \qquad \text{for} \quad * > n\,.$$

Now use 2.2 to conclude that D_H is $(-n-1)$-connected.

By 2.5, there is an unequivariant weak equivalence

$$D_H \simeq D_G\,.$$

It follows that D_G is also $(-n-1)$-connected. But then so is the spectrum $D_G \wedge_{hG} E$ since E is (-1)-connected. Using the homotopy cofiber sequence

$$D_G \wedge_{hG} E \to E^{hG} \to E^{tG}$$

one infers that the map $E^{hG} \to E^{tG}$ is $(-n)$-connected. One concludes from this that $\widehat{E}^*(G)$ and $E^*(G)$ are isomorphic in degrees $* > n$. □

10. Appendix: Examples

1. BG is a finitely dominated Poincaré duality space. According to 5.1, $\Sigma^j D_G \simeq_G \Sigma^\infty Y$ where Y is unequivariantly a sphere. Moreover, the Borel construction $Y \to EG\times_G Y \to BG$ gives the Spivak fibration.

Hence, if BG has dimension n, there is an equivariant weak equivalence

$$D_G \simeq_G S^{\nu[-n]}\,,$$

where $S^{\nu[-n]}$ is the fiber of the Spivak fibration desuspended down to degree $-n$.

2. The case of a compact Lie group.

Theorem 10.1. *Assume that G is a compact Lie group. Then there is an equivariant weak equivalence*

$$D_G \simeq_G S^{\mathrm{Ad}_G}\,,$$

where the right side denotes the suspension spectrum of the one point compactification of the adjoint representation of G.

Proof of 10.5 (Sketch). Thinking unstably, for the moment we take S^{Ad_G} to mean the one point compactification of the Lie algebra $\mathfrak{g}$ of G, with the $G\times G$ action on it in which $G\times 1$ acts trivially and $1\times G$ acts via the adjoint representation $\mathrm{Ad}_G\colon G \to \mathrm{GL}(\mathfrak{g})$. Give G_+ the action of $G\times G$ defined by $(g,h)*x = gxh^{-1}$. Give the based function space $F(G_+, S^{\mathrm{Ad}_G})$ the action of $G\times G$ defined by conjugation of functions: $(g,h)*\phi(y) = \mathrm{Ad}_G(h)(\phi(g^{-1}yh))$.

Let

$$\log\colon G \to S^{\mathrm{Ad}_G}$$

be defined as follows: choose $\epsilon > 0$ such that the exponential map $\exp\colon \mathfrak{g} \to G$ is an embedding on $D(\epsilon) =$ the disk of radius ϵ. Identify S^{Ad_G} with $D(\epsilon)/\partial D(\epsilon)$. Define $\log(x)$ to be z if $\exp(z) = x$ and z has norm $\le \epsilon$, and ∞ otherwise.

Then the map

$$\alpha\colon G_+ \to F(G_+, S^{\mathrm{Ad}_G})$$

given by $\alpha(x)(y) := \log(x^{-1}y)$ is $(G\times G)$-equivariant. (This uses the fact that $g\exp(x)g^{-1} = \exp^{\mathrm{Ad}_G(g)(x)}$ for all $g \in G$, $x \in \mathfrak{g}$.)

The adjunction map $\hat{\alpha}\colon G_+ \wedge G_+ \to S^{\mathrm{Ad}_G}$ of α is a Spanier-Whitehead duality. (Reason: using the trivialization of the tangent bundle of G given by left translation, $F(G_+, S^{\mathrm{Ad}_G})$ is identified with the space of sections of the fiberwise one point compactification of the tangent bundle. With respect to this identification, the map α gives the tangential version of Atiyah duality [At].)

Passing to the stable category, we infer that α induces a $(G\times G)$-equivariant weak equivalence of spectra

$$S^0[G] \simeq_{G\times G} F(G_+, S^{\mathrm{Ad}_G})\,.$$

Taking homotopy fixed sets with respect to $1\times G$, we obtain

$$D_G = (S^0[G])^{h(G\times 1)} \simeq_G F(G_+, S^{\mathrm{Ad}_G})^{G\times 1} = S^{\mathrm{Ad}_G}\,.$$

Corollary 10.2. *Suppose that G is a compact Lie group and that W is an unequivariant spectrum. Let $E = W \wedge G_+$. Then the norm map $D_G \wedge_{hG} E \to E^{hG}$ is a weak equivalence.*

Proof. The proof of 10.1 shows that $S^0[G]$ and $F(G_+, S^{\mathrm{Ad}_G})$ are $(G\times G)$-equivariantly weak equivalent. Consequently, smashing with W we get

$$W \wedge G_+ \simeq_{G\times G} W \wedge F(G_+, S^{\mathrm{Ad}_G})\,.$$

Since G_+ is a finite complex, the small object argument implies that

$$W \wedge F(G_+, S^{\mathrm{Ad}_G}) \simeq_{G\times G} F(G_+, S^{\mathrm{Ad}_G} \wedge W)\,.$$

Therefore we get a weak equivalence

$$W \wedge G_+ \simeq_{G\times G} F(G_+, S^{\mathrm{Ad}_G} \wedge W)\,.$$

Taking homotopy fixed sets with respect to $G\times 1$, we get

$$E^{hG} = (W \wedge G_+)^{hG} \simeq_G S^{\mathrm{Ad}_G} \wedge W \simeq_G D_G \wedge W \simeq_G D_G \wedge_{hG} E\,.$$

A (tedious) check which we omit shows that this identification coincides with the norm map up to homotopy. □

3. G **is a finitely dominated topological group.** By 2.5, we can assume that G is connected. We have an extension

$$\Omega G \to PG \to G$$

given by the path fibration. By Theorem H, $B\Omega G \simeq G$ is a Poincaré duality space, so Theorem B says that $D_{\Omega G} \wedge D_G \simeq D_{PG} = S^0$. If G has dimension n as a Poincaré duality space, then $D_{\Omega G} \simeq S^{-n}$. It follows that there is an unequivariant weak equivalence

$$D_G \simeq S^n\,.$$

4. Γ **is a torsion free discrete cocompact subgroup of a connected Lie group** G**.** In this instance $B\Gamma$ is homotopy equivalent to the compact closed manifold

$$\Gamma\backslash G/K$$

where $K \subset G$ is any maximal compact subgroup. So by Theorem A, there is a weak equivalence

$$D_\Gamma \simeq S^{k-n}$$

where $n = \dim G$ and $k = \dim K$.

5. Γ **is a finitely generated free group.** Suppose that Γ is a free group on g generators. Let H_g be a handlebody of genus g embedded in $\mathbb{R}^3$. Then $B\Gamma \simeq H_g$, and 10.5 and 10.6 below show

$$D_\Gamma \simeq_\Gamma S^{-3} \wedge \Sigma^{\mathrm{u}} B\pi_g\,,$$

where Σ^{u} denotes unreduced suspension and $B\pi_g$ is the space with Γ-action defined as follows: let π_g be the kernel of the homomorphism $\pi_1(\partial H_g) \to \pi_1(H_g)$. Then π_g acts freely on the universal cover of the surface ∂H_g. The universal cover is contractible, so a model for the classifying space $B\pi_g$ is given by taking the orbit space of the π_g-action. The orbit space therefore inherits a Γ-action.

Unequivariantly, it is elementary to check that D_Γ weak equivalent to an infinite countable wedge of (-1)-spheres.

Remark 10.3. Since finitely generated free groups are arithmetic, one can alternatively identify the dualizing spectrum in this case by appealing to example 8 below.

6. $\Gamma = P_n$ is the pure braid group. Recall that P_n is defined to be the fundamental group of the ordered configuration space of n points in $\mathbb{R}^2$. The latter is an Eilenberg-Mac Lane space, so BP_n is in particular homotopy finite.

Forgetting the last point in a configuration defines an extension

$$1 \to \mathbb{Z}^{*g} \to P_n \to P_{n-1} \to 1\,.$$

Iterated application of Theorem B now shows that

$$D_{P_n} \simeq D_{\mathbb{Z}^{*g}} \wedge D_{\mathbb{Z}^{*(g-1)}} \wedge \cdots \wedge D_{\mathbb{Z}}\,.$$

Each factor on the right side is weak equivalent to a countably infinite wedge of (-1)-spheres. Consequently, D_{P_n} is unequivariantly weak equivalent to a countably infinite wedge of spheres of dimension $\frac{-1}{2}g(g+1)$.

7. $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2$ is the infinite dihedral group. It is well-known that the infinite dihedral group has an infinite cyclic (normal) subgroup of index 2. Consequently 2.5 shows that

$$D_{\mathbb{Z}_2 * \mathbb{Z}_2} \simeq D_{\mathbb{Z}} \simeq S^{-1}$$

(the last equality is a consequence of Theorem A and the fact that $B\mathbb{Z} = S^1$).

8. Γ is a torsion free arithmetic group. It is known by work of Borel and Serre [B–S] that there is a model for $B\Gamma$ which can be compactified to a compact manifold with corners.

Namely, the space $X := G(\mathbb{R})/K$ is a model for $E\Gamma$, where $G(\mathbb{R})$ is the group of real points of the algebraic group where Γ lives, and K is a choice of maximal compact subgroup. Borel and Serre define a manifold with (free) Γ-action $\bar{X}$ by adding corners to X in a suitable way. The compactification of $B\Gamma$ is then

$$\bar{Y} = \bar{X}/\Gamma\,.$$

The space $\bar{X}$ is gotten from X by adjoining a 'partial' boundary $\partial\bar{X}$ which has the Γ-equivariant homotopy type of $\Delta =$ the Solomon-Tits building of the group of rational points of G (see [Br3, Chap. 7] for more details).

Let S^τ denote the Thom space of the tangent bundle of $\bar{X}$. As $\bar{X}$ is contractible, S^τ is a sphere having the same dimension as $\bar{X}$. Moreover, S^τ comes equipped with a based Γ-action. Then unreduced Borel construction $S^\tau \to S^\tau \times_\Gamma \bar{X} \to \bar{Y}$ has the fiber homotopy type of the fiberwise one point compactification of the tangent bundle of $\bar{Y}$. Using 10.5 and 10.6 below, we infer

Theorem 10.4. *There is an equivariant weak equivalence*

$$D_\Gamma \;\simeq_\Gamma\; F(S^\tau, \Sigma^\infty \Sigma^{\mathrm{u}} \Delta)\,,$$

where $\Sigma^{\mathrm{u}}\Delta$ is the unreduced suspension of the Solomon-Tits building Δ. In particular, up to an orientation character, the homology of D_Γ coincides with the Steinberg representation.

Since Δ is homotopy equivalent to wedge of spheres, unequivariantly, D_Γ is a wedge of spheres.

Examples 5 and 8 made use of the following result:

Proposition 10.5. *Let G be a topological group. Assume that BG comes equipped with a weak equivalence $h\colon Y \xrightarrow{\sim} BG$ in which $(Y, \partial Y)$ is a finitely dominated Poincaré pair. Let S^τ denote the the fiber to Spivak tangent fibration of Y (dimension shifted so that it has degree n), considered as a G-spectrum. Then there is an equivariant weak equivalence*

$$D_G \simeq_G F(S^\tau, \Sigma^\infty \widetilde{Y}/\partial\widetilde{Y}),$$

where $(\widetilde{Y}, \partial\widetilde{Y})$ denotes the fiber product $(Y \times^{BG} EG, \partial Y \times^{BG} EG)$.

Remark 10.6. Note that $\widetilde{Y}$ is weakly contractible, so $\widetilde{Y}/\partial\widetilde{Y}$ is equivariantly weak equivalent to $\Sigma^{\mathrm{u}}\partial\widetilde{Y}$, the unreduced suspension of $\partial\widetilde{Y}$.

Proof of 10.5 (Sketch). Let $p\colon (E, E_{|\partial Y}) \to (Y, \partial Y)$ be the Spivak normal fibration. By taking fiberwise join with S^0 if necessary, we can assume that p comes equipped with a section. The characterizing property of the Spivak fibration is that comes equipped with a map

$$\alpha\colon S^j \to E^\nu/E^\nu_{|\partial Y}$$

whose target is the Thom space of p, in which the cap product of $\alpha_*([S^j])$ with the Thom class of p is a fundamental class for $(Y, \partial Y)$.

Up to fiber homotopy equivalence, we can rewrite p as a Borel construction

$$(S^\nu \times_G \widetilde{Y}, S^\nu \times_G \partial\widetilde{Y}) \to (Y, \partial Y),$$

where S^ν represents the fiber of p together with its based G-action.

With respect to this identification, the Thom space $E^\nu/E^\nu_{|\partial Y}$ is identified with $S^\nu \wedge_{hG} \widetilde{Y}/\partial\widetilde{Y}$. In this representation, α becomes a map

$$\beta\colon S^j \to EG_+ \wedge_G (S^\nu \wedge \widetilde{Y}/\partial\widetilde{Y}),$$

and the relation between α, the Thom isomorphism and Poincaré duality translates to the statement that β is an equivariant duality map (this also uses [Kl5, Prop. 6.4]).

On the other hand, as in the proof of $3 \Rightarrow 1$ of Theorem A (see Sect. 6), we know that there exist an integer $k \gg 0$, an equivariant weak equivalence

$$S^j \wedge D_G \simeq_G \Sigma^\infty Z$$

and an equivariant duality map $S^k \to EG_+ \wedge_G Z$. By suspending if necessary, we can assume that $k = j$.

By the uniqueness theorem for equivariant duals [Kl5, Thm 6.5], we may conclude that there is an equivariant weak equivalence

$$(S^\nu \wedge \widetilde{Y}/\partial\widetilde{Y}) \simeq_G \Sigma^\infty Z\,.$$

Consequently, there is an equivariant weak equivalence

$$D_G \simeq_G S^\nu \wedge \widetilde{Y}/\partial\widetilde{Y}\,.$$

Since $S^\tau \wedge S^\nu \simeq_G S^0$, we have an identification $S^\nu \simeq_G F(S^\tau, S^0)$. Since S^τ is G-finitely dominated, we have $F(S^\tau, S^0) \wedge E \simeq_G F(S^\tau, E)$ for any G-spectrum E. In particular,

$$D_G \simeq_G S^\nu \wedge \widetilde{Y}/\partial\widetilde{Y} \simeq_G F(S^\tau, \Sigma^\infty\widetilde{Y}/\partial\widetilde{Y})\,.$$

□

9. Γ is a Bieri-Eckmann duality group. Assume that $B\Gamma$ is finitely dominated. Recall that Γ is a *duality group* of dimension n if there exists a $\mathbb{Z}[\Gamma]$-module D such that in every degree there is an isomorphism

$$H^*(\Gamma; M) \cong H_{n-*}(\Gamma; D \otimes_{\mathbb{Z}} M).$$

for any $\mathbb{Z}[\Gamma]$-module M (see [Br4, Chap. 8, Sect. 10] for the basic properties of duality groups). The module D is called the *dualizing module* of Γ, and is isomorphic to $H^n(\Gamma; \mathbb{Z}[\Gamma])$. It is known that D is torsion-free as an abelian group. If D is finitely generated and has rank one, then $B\Gamma$ is a Poincaré space, and in this instance one says that Γ is a *Poincaré duality group.*

The following result characterizes the dualizing spectra of duality groups. We omit the proof, since it essentially follows along the lines of the proof of Theorem A.

Theorem 10.7. *A group Γ is a duality group (of dimension n) if and only if its dualizing spectrum D_Γ is unequivariantly weak equivalent to a Moore spectrum in degree $-n$ on a torsion free abelian group.*

(Recall that a *Moore spectrum* in degree j on an abelian group A is a spectrum Y whose spectrum homology $\pi_*(Y \wedge H\mathbb{Z})$ vanishes except in dimension j, and whose homology in degree j is isomorphic to A.)

Remark 10.8. If Γ is a duality group of dimension n, then it is not difficult to see that the spectrum homology of D_Γ in degree $-n$ coincides with the dualizing module of Γ.

If D_Γ is not a $(-n)$-sphere, then it follows from Theorem A that D_Γ is not a homotopy finite spectrum. We infer that D_Γ is a Moore spectrum on an abelian group which is not finitely generated. Thus, we recover a result of Farrell [Fa] which says that *the dualizing module of a duality group is finitely generated if and only if the group is a Poincaré duality group (i.e., the dualizing module has rank one).*

10. The case $\Gamma = \mathbb{Z}^d * \mathbb{Z}^m$. Let us call a diagram

$$\begin{array}{ccc} R & \longrightarrow & Q \\ \downarrow & & \downarrow \\ P & \longrightarrow & G \end{array}$$

of (topological or discrete) groups an *amalgamation diagram* if it becomes homotopy cocartesian after applying the classifying space functor. Associated to an amalgamation diagram, there is a homotopy cartesian square of spectra

$$\begin{array}{ccc} D_G & \longrightarrow & (S^0[G])^{hQ} \\ \downarrow & & \downarrow \\ (S^0[G])^{hP} & \longrightarrow & (S^0[G])^{hR} . \end{array}$$

We apply this in the following situation: let R be the trivial group, $P = \mathbb{Z}^d$ and $Q = \mathbb{Z}^m$ with $d, m > 0$. Since P and Q are Poincaré duality groups, the square in this case becomes

$$\begin{array}{ccc} D_G & \longrightarrow & S^{-m} \wedge_P G_+ \\ \downarrow & & \downarrow \\ S^{-d} \wedge_P G_+ & \longrightarrow & S^0[G] , \end{array}$$

where we have used Theorem D to rewrite the lower left and upper right hand corner as homotopy orbits. Note that the action of P on D_P is trivial. Consequently $S^{-d} \wedge_P G_+$ is an countably infinite wedge of copies of S^{-d}. Similarly, $S^{-m} \wedge_P G_+$ is a countably infinite wedge of copies of S^{-m}, while $S^0[G]$ is a countably infinite wedge of copies of S^0. For dimensional reasons, the maps in the diagram are null homotopic. Consequently,

$$D_{\mathbb{Z}^d * \mathbb{Z}^m} \simeq \bigvee_I (S^{-1} \vee S^{-d} \vee S^{-m})$$

where I is a countably infinite indexing set. In particular, $\mathbb{Z}^d * \mathbb{Z}^m$ is *not* a duality group unless $d = m = 1$.

A problem, a question and a conjecture. In all examples above, D_G turned out to be unequivariantly weak equivalent to a wedge of spheres. It would be interesting to have other kinds of examples, especially in the case of a discrete group.

Based on the technique of "hyperbolization" [D–J], Bestvina and Mess [B–M] have given examples of discrete groups Γ such that $B\Gamma$ is homotopy finite and $H^3(\Gamma; \mathbb{Z}[\Gamma]) \cong \mathbb{Z}/2$. This implies that D_Γ is not the homotopy type of a wedge of spheres.

Problem. *Compute the homotopy type of D_Γ in the Bestvina-Mess examples.*

We know that Bieri-Eckmann duality groups Γ are such that D_Γ has the unequivariant weak homotopy type of a Moore spectrum on a torsion free abelian group.

Question. *In the case of a duality group Γ, is the dualizing spectrum always a wedge of spheres? (This is a rephrasing of the old question which asks whether the dualizing module is free abelian.)*

Finally, there is the issue of whether or not the unequivariant homotopy type of the dualizing spectrum is a *coarse* invariant. If Γ is a finitely generated group, then the word metric equips Γ with the structure of a metric space.

Conjecture. *Suppose that Γ and Γ' are quasi-isometric.*[3] *Assume that Γ and Γ' have homotopy finite classifying spaces. Then D_Γ and $D_{\Gamma'}$ are unequivariantly weak equivalent.*

There is positive evidence for this conjecture: with respect to our assumptions, the spectrum homology of D_Γ coincides with $H^*_f(E\Gamma;\mathbb{Z})$, the cohomology of $E\Gamma$ with *finite supports*. Gersten [Ge, Th. 8] has shown that that $H^*_f(E\Gamma;\mathbb{Z})$ and $H^*_f(E\Gamma';\mathbb{Z})$ are isomorphic. Consequently, the spectrum homology of D_Γ and $D_{\Gamma'}$ are isomorphic.

References

[A–C–D] Adem, A., Cohen, R., Dwyer, W.: Generalized Tate homology, homotopy fixed points and the transfer. In: Algebraic Topology (Evanston 1988) Contemp. Math. **96**, pp. 1–13. AMS 1989

[At] Atiyah, M.: Thom complexes. Proc. London Math. Soc. **11**, 291– 310 (1961)

[B–K] Bousfield, A. K., Kan, D. M.: Homotopy Limits, Completions and Local izations. (LNM, Vol. 304). Springer 1972

[B–M] Bestvina, M., Mess, G.: The boundary of negatively curved groups. JAMS **4**, 469–481 (1991)

[B–S] Borel, A., Serre, J.–P.: Corners and arithmetic groups. Comment. Math. Helv. **48**, 244–297 (1974)

[Br1] Browder, W.: Surgery an Simply Connected Manifolds. (Ergebnisse Math., Vol. 65). Springer 1972

[Br2] Browder, W.: Torsion in H-spaces. Ann. Math. **74**, 24–51 (1961)

[Br3] Brown, K.: Buildings. Springer 1989

[Br4] Brown, K.: Cohomology of Groups. (GTM, Vol. 87). Springer 1982

[D–J] Davis, M., Januszkiewicz, T.: Hyperbolization of polyhedra. J. Differential Geom. **34**, 347–388 (1991)

[Fa] Farrell, F.: Poincaré duality and groups of type (FP). Comment. Math. Helv. **50**, 187–195 (1975)

[3] See e.g., [Ge].

[G–M] Greenlees, J., May, J.: Generalized Tate Cohomology. (AMS Memoirs, Vol. 543). AMS 1995
[Ge] Gersten, S. M.: Isoperimetric functions of groups and exotic cohomology. In: Combinatorial and geometric group theory (Edinburgh, 1993), LMS Lecture Notes **204**, pp. 87–104. Cambridge Univ. Press 1995
[Go] Gottlieb, D. H.: Poincaré duality and fibrations. Proceedings AMS **76**, 148–150 (1979)
[Kl1] Klein, J. R.: An axiomatic characterization of Farrell–Tate cohomology. Preprint
[Kl2] Klein, J. R.: On two results about fibrations. Manuscripta Math. **92**, 77–86 (1997)
[Kl3] Klein, J. R.: Poincaré duality spaces. In: Surveys in Surgery Theory: papers dedicated to C. T. C. Wall, Annals of Math. Studies. Princeton Univ. Press. To appear
[Kl4] Klein, J. R.: Poincaré embeddings and fiberwise homotopy theory. Topology **38**, 597–620 (1999)
[Kl5] Klein, J. R.: Poincaré immersions. Forum Math. **11**, 717–734 (1999)
[Qu1] Quillen, D.: Homotopical Algebra. (LNM, Vol. 43). Springer 1967
[Qu2] Quinn, F.: Surgery on Poincaré spaces. Unpublished Manuscript, 1971
[Sc] Schwede, S.: Spectra in model categories and applications to the algebraic cotangent complex. J. Pure Appl. Algebra **120**, 77–104 (1997)
[Wa1] Wall, C. T. C.: Finiteness conditions for CW–complexes. II. Proc. Roy. Soc. Ser. A **295**, 129–139 (1966)
[Wa2] Wall, C. T. C.: Poincaré complexes I. Ann. of Math. **86**, 213–245 (1967)
[Wa3] Wall, C. T. C.: Surgery an Compact Manifolds, Academic Press 1970