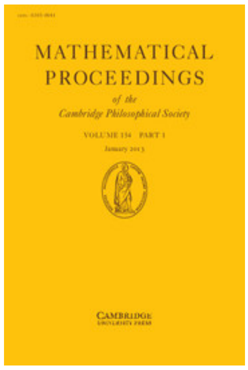


# Structure theorems for homotopy pushouts I: contractible pushouts

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# Structure theorems for homotopy pushouts I: contractible pushouts

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*Abstract*

We define a classifying space for contractible homotopy pushout diagrams and then study its homotopy type.

*Introduction*

The purpose of this series of papers is to study homotopy pushout squares, i.e., commutative diagrams of spaces of the form

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ K & \longrightarrow & X \end{array} \tag{0·1}$$

such that the canonical map

$$K \cup_{A\times 0} A \times I \cup_{A\times 1} C \to X$$

is a homotopy equivalence, where the source denotes the double mapping cylinder of the diagram $K \leftarrow A \to C$. In this paper, we will only consider the case $X = \mathrm{pt}$; we may then specify the diagram (0.1) in a compact way as a map

$$A \to K \times C\,.$$

Observe that $A$ homologically behaves as the wedge $K \vee C$. In fact, $A$ splits after one suspension as the wedge of $K$ and $C$. However, the following elementary example shows that $A$ might not split unstably.

*Example* 0·2. Let $K = S^m$ and $C = S^m_+ \wedge S^m \simeq S^{2m} \vee S^m$, with $m \geqslant 1$. Let $A = S^m \times S^m$, with $A \to K$ the projection onto the first factor, and $A \to C$ the map which collapses $S^m \times * \subset S^m \times S^m$ to a point. Then the data describe a contractible homotopy pushout. It is easy to check by means of cohomology rings that $A$ fails to split as $K \vee C$.

Now assume that $K$ and $C$ denote 1-connected, based spaces. Let $\mathfrak{D}(K, C)$ denote the category in which an object is a contractible homotopy pushout diagram of based spaces $A \to K \times C$ such that $A$ is also 1-connected. A morphism from an object $A \to K \times C$ to an object $A' \to K \times C$ is specified by a based homotopy equivalence $A \to A'$ which commutes with projection to $K \times C$. Let

$$\mathscr{D}(K, C) := |\mathfrak{D}(K, C)|$$

denote the classifying space of $\mathfrak{D}(K, C)$, that is, the geometric realization of its nerve.[1] Thus, $\mathscr{D}(K, C)$ is a kind of classifying space for contractible homotopy pushouts with vertices $K$ and $C$. Note that $\mathscr{D}(K, C)$ is equipped with a basepoint given by the 'trivial' diagram $K \vee C \to K \times C$.

Suppose that $K$ is homotopy equivalent to a CW complex of dimension $k$ but not to one of $k - 1$. In this instance we write $\dim_{CW} K = k$. Similarly, let $\dim_{CW} C = c$. Let $r = \operatorname{conn} K$ be the connectivity of $K$, and let $s = \operatorname{conn} C$ denote the connectivity of $C$. For based spaces $Y$ and $Z$, let $\{Y, Z\}$ denote the abelian group of homotopy classes of stable maps from $Y$ to $Z$.

Our first result is

THEOREM A. (1) *There is a homotopy equivalence*

$$\mathscr{D}(K, C) \simeq \coprod_{\alpha} BM_{\alpha}$$

*where $M_{\alpha}$ is a certain group-like topological monoid (defined in 3·1 below), $BM_{\alpha}$ denotes its classifying space and the disjoint union is indexed over the elements of $\pi_0(\mathscr{D}(K, C))$.*

(2) *There is a map of pointed sets*

$$\pi_0(\mathscr{D}(K, C)) \to \{K \vee C, K \wedge C\}$$

*which is surjective if $k, c \leqslant 3 \min(r, s)$ and an isomorphism if $k, c \leqslant 3 \min(r, s) - 1$.*

Without knowledge of the explicit form of the topological monoids $M_{\alpha}$, the first part of Theorem A has trivial content: any space decomposes as a disjoint union of its connected components. By choosing a basepoint, each connected component may be written up to weak homotopy equivalence as the classifying space of its associated (Moore) loop space (the latter a topological monoid). The interest in the first part of Theorem A lies in the explicit description of the $M_{\alpha}$. In fact, $M_{\alpha}$ can be defined as the topological monoid of homotopy automorphisms of a choice of basepoint in the $\alpha$-component of $\mathscr{D}(K, C)$. This description will be used in proving Theorems B and C (see below), and it is for this reason that we state the first part.

Here is an illustration of the second part of Theorem A, with $K = S^m = C$. The result says that there is an isomorphism of pointed sets

$$\pi_0(\mathscr{D}(S^m, S^m_+ \wedge S^m)) \cong \{S^m \vee S^m_+ \wedge S^m, S^m \wedge S^m_+ \wedge S^m\} = \mathbb{Z}\,.$$

The example of 0·2 will be seen to correspond to $1 \in \mathbb{Z}$ under this isomorphism. In general, to represent the other integers, we let $A$ be the result of attaching an integral multiple $\ell$ of the Whitehead product map $S^{2m-1} \to S^m \vee S^m$. This yields a contractible pushout diagram $A \to S^m \times (S^m_+ \wedge S^m)$ which corresponds to the integer $\ell$ via the isomorphism.

[1] As usual, in order to make sense of geometric realization, we have to replace this category with a suitably 'small' model, by restricting what one means by the term 'space'. For our purposes 'space' will denote a compact ANR embedded in $\mathbb{R}^{\infty}$. Alternatively, one can work with countable CW complexes. In the end, the model to be used is basically a matter of taste.

For certain applications, one wants to allow $K$ and $C$ to vary within their homotopy types. For this purpose, we define $\mathfrak{E}(K,C)$ to be the category whose objects are contractible homotopy pushout diagrams $A' \to K' \times C'$ where $K'$ and $C'$ are (abstractly) homotopy equivalent to $K$ and $C$ respectively, and $A'$ is 1-connected. A morphism

$$(A' \to K' \times C') \longmapsto (A'' \to K'' \times C'')$$

is specified by homotopy equivalences $A' \to A''$, $K' \to K''$ and $C' \to C''$ such that the evident diagram is commutative. Set $\mathscr{E}(K,C) = |\mathfrak{E}(K,C)|$.

For a based space $X$, let $G(X)$ denote the topological monoid of based homotopy automorphisms of $X$. Our second result identifies the homotopy type of $\mathscr{E}(K,C)$ in terms of $\mathscr{D}(K,C)$, $G(K)$ and $G(C)$:

THEOREM B. *There is a fibration up to homotopy*

$$\mathscr{D}(K,C) \to \mathscr{E}(K,C) \to BG(K) \times BG(C)\,.$$

*Moreover, this homotopy fibration admits a section. Consequently,* $\pi_0(\mathscr{D}(K,C)) = \pi_0(\mathscr{E}(K,C))$ *and there is a homotopy equivalence*

$$\Omega\mathscr{E}(K,C) \simeq \Omega\mathscr{D}(K,C) \times G(K) \times G(C)\,.$$

By combining Theorem B with Theorem A, we obtain a homotopy equivalence

$$\Omega\mathscr{E}(K,C) \simeq M_0 \times G(K) \times G(C)\,,$$

where $M_0$ is the topological monoid (of 3·1 below) which corresponds to the component of the trivial diagram $K \vee C \to K \times C$.

Our third result is a kind of stability theorem for contractible homotopy pushouts. Given a pushout square of the kind 0·1, again with $X = *$, we can form a new pushout

$$\begin{array}{ccc} \Sigma_C A & \longrightarrow & C \\ \big\downarrow & & \big\downarrow \\ \Sigma K & \longrightarrow & * \end{array} \tag{0·2}$$

where $\Sigma_C A$ denotes the *fibrewise suspension* of $A$ over $C$; it is defined by taking the homotopy colimit of the diagram $C \leftarrow A \to C$. It naturally maps to $\Sigma K$ (the suspension of $K$). This modified diagram has more structure than the preceding one: There is a cofibration $C \vee C \to \Sigma_C A$ such that the composition $C \vee C \to \Sigma_C A \to C$ coincides with the fold map, and the composite map $C \vee C \to \Sigma_C A \to \Sigma K$ is the constant map to the basepoint.

Therefore, let $\mathfrak{D}^{sc}(K,C)$ be the category whose *objects* are contractible homotopy pushouts $A \to K \times C$ which are equipped with a cofibration $C^{\vee 2} \to A$ such that the diagram

$$\begin{array}{ccc} C^{\vee 2} & \longrightarrow & A \\ {\scriptstyle \text{fold}}\big\downarrow & & \big\downarrow \\ C & \xrightarrow[*\times \mathrm{id}_C]{} & K \times C \end{array}$$

is commutative. A *morphism* $A \to A'$ is a map which commutes with projection to $K \times C$ and also commutes with the cofibration data. Set $\mathscr{D}^{sc}(K, C) := |\mathfrak{D}^{sc}(K, C)|$.

The construction described above gives rise to a map

$$\sigma_C : \mathscr{D}(K, C) \to \mathscr{D}^{sc}(\Sigma K, C)\,.$$

Our third result identifies the connectivity of this map in terms of $\dim_{CW} K = k$, $\dim_{CW} C = c$, $\mathrm{conn}\,(K) = r$ and $\mathrm{conn}\,(C) = s$.

THEOREM C. *The map*

$$\sigma_C : \mathscr{D}(K, C) \to \mathscr{D}^{sc}(\Sigma K, C)$$

*is* $(2r + s - \max\,(k, c) + 2)$*-connected.*

Note that the range where $\sigma_C$ induces a bijection of components is typically larger than the range stated in Theorem A(2). In a future paper, we will show how to generalize Theorem C to a wider range (roughly $2(r + s) - \max\,(k, c)$, an additional gain of about $r$). This will be important for geometric applications.

Contractible pushouts naturally arise from polyhedra embedded in euclidean space. For instance, let $Y \subset \mathbb{R}^n$ be a finite polyhedron. Let $K$ denote a closed regular neighborhood of $Y$, $C$ the closure of its complement and $A = \partial K = \partial C$. Then $A$ maps to both $K$ and $C$ with $\mathbb{R}^n$ as the pushout.

This example is the real motivation for writing this paper. In fact, homotopy theoretic information about spaces of pushouts has implications for the theory of Poincaré embeddings. The connection is evident from the definition: A Poincaré embedding from a Poincaré space $K$ to a Poincaré space $X$ is just a pushout diagram involving $K$ and having pushout $X$, plus additional duality constraints. I will be using the results/methods of this series of papers to classify Poincaré embeddings in a wide range (see [**4**] for a restricted case of this programme in euclidean space).

0·4. *Outline.* Section 1 is language. In Section 2 we prove an approximation result for homotopy pushout diagrams which says, roughly, that if we are given a pushout square in which the terminal vertex is highly connected, then the square may be replaced by another pushout square in which the terminal vertex is contractible. In Section 3 we prove the first part of Theorem A; the proof is basically a modification of a result of Waldhausen. In Section 4 we prove the second part of Theorem A. The main tool for the proof is the EHP-sequence for stable homotopy. In Section 5 we prove Theorem B; its proof relies on Quillen's 'Theorem B'. In Section 6 we prove Theorem C. Here, the main tool will be a combination of Theorem A(1) together with Goodwillie's generalized Blakers–Massey theorem and its dual for $n$-dimensional cubes of spaces.

## 1. *Conventions*

Let **Top** denote the category of compactly generated spaces, and let $\mathbf{Top}_*$ denote the category of compactly generated, based spaces. Unless otherwise stated, the spaces of this paper which appear as single upper case letters $A, B, C, \ldots$ (and also their descendents $A', B_1, \ldots$ etc.) will always be objects of $\mathbf{Top}_*$ which are equipped with non-degenerate basepoints. Unless otherwise stated, they will also have the homotopy type of a *finite* CW complex. In particular, the objects of the categories $\mathfrak{D}(K, C)$, $\mathfrak{E}(K, C)$ etc., of the introduction are defined using 1-connected, non-degenerately based spaces which have the homotopy type of a finite CW complex. Function spaces are to be given the compact-open topology. The term *cofibration* means a closed inclusion which satisfies the homotopy extension property. This paper uses facts about homotopy colimits and homotopy limits in $\mathbf{Top}_*$. A standard reference for homotopy limits and colimits is [**1**] (who work instead in the category of based simplicial sets; for a treatment in $\mathbf{Top}_*$, see [**3**]).

We will be observing the following (standard) connectivity conventions: A map $X \to Y$ of unbased spaces is said to be 0-*connected* if $\pi_0(X)$ maps onto $\pi_0(Y)$. It is called $r$-*connected*, for $r > 0$, if $\pi_0(X)$ maps bijectively onto $\pi_0(Y)$ and for all basepoints in $X$, the map $\pi_i(X) \to \pi_i(Y)$ is a surjection for $i \leqslant r$ and an isomorphism for $i < r$. Every map is $(-1)$-connected.

## 2. *Converting highly connected pushouts into contractible ones*

Let

$$K \leftarrow B \rightarrow C$$

be a diagram of 1-connected, based spaces. As in the introduction, we set $\dim_{CW} K = k$ and $\dim_{CW} C = c$. Let $P$ denote the homotopy pushout of the diagram.

LEMMA 2·1. *Suppose that the homotopy pushout of the diagram is $j$-connected, where $k, c \leqslant j$. Then there exists a space $A$ and a $(j-1)$-connected map $A \to B$ such that the homotopy pushout of the associated diagram*

$$K \leftarrow A \rightarrow C$$

*is contractible.*

*Proof.* If $j = 0, 1$, then $K$ and $C$ are automatically contractible and we can take $A$ to be a point. So for the rest of the proof, we may take $j \geqslant 2$.

Without loss in generality we may assume that $B$ is a CW complex, and that the maps $B \to K$ and $B \to C$ are cofibrations. Denote these maps by $u$ and $u'$ respectively. Let $B^t$ denote the $t$-skeleton of $B$. Then the map $B^j \to B$ is $j$-connected, and the Mayer–Vietoris sequences imply that

$$K \cup_{B^j} C$$

is also $j$-connected. So without any further loss in generality, we can assume that $B = B^j$.

It follows that there is a short exact sequence

$$0 \to H_{j+1}(K \cup_B C) \to H_j(B) \xrightarrow{u_* \oplus u'_*} H_j(K) \oplus H_j(C) \to 0\,,$$

and we have $H_*(K \cup_B C) = 0$ for $* \neq j, j+1$. Moreover, the sequence splits, since $H_j(K) \oplus H_j(C)$ is free abelian. The sequence shows that we have to kill the kernel of the map $u_* \oplus u'_*\colon\ H_j(B) \to H_j(K) \oplus H_j(C)$.

Since $B$ is $j$-dimensional, there is an injection

$$H_j(B) \overset{\subset}{\longrightarrow} H_j(B^j, B^{j-1}) \cong \pi_j(B^j, B^{j-1})\,.$$

Furthermore, $H_j(B^j, B^{j-1}) \cong H_j(K \cup_B C) \oplus T$, for some free module $T$. Choose a basis $\{\alpha_i\}_{i\in I}$ for $T$. Represent $\alpha_i$ as a map $\alpha_i\colon (D^j, S^{j-1}) \to (B^j, B^{j-1})$, using the above injection. Let

$$\alpha\colon \bigvee_{i\in I}(D^j, S^{j-1}) \longrightarrow (B^j, B^{j-1})$$

denote the union of the maps $\alpha_i$ for $i \in I$.

Let $A$ be given by

$$B^{j-1} \cup_\alpha \bigvee_{i\in I} D^j\,.$$

Then we obtain a $(j-1)$-connected map $A \to B$, and it is straightforward to check that the homotopy pushout of $K \leftarrow A \to C$ is contractible. Now if $j > 2$, it is automatic that $A$ is 1-connected. This establishes the result when $j > 2$.

When $j = 2$, then $A$ might not be simply connected, however, the first homology of $A$ is certainly trivial. Taking Quillen's plus construction in this case yields a simply connected space $A^+$ and a factorization $A \to A^+ \to B$. Then $A^+ \to B$ satisfies the conclusion of the lemma.

*Definition* 2·2. Let $j$ be a positive integer or infinity. Define a category $\mathfrak{D}^j(K, C)$ whose *objects* are specified by a map of 1-connected, based spaces $B \to K \times C$ such that the homotopy pushout of the associated diagram $K \leftarrow B \to C$ is $j$-connected. A *morphism* $(B \to K \times C) \to (B' \to K \times C)$ is a ($(j-1)$-connected) map $B \to B'$ which commutes with the structure map to $K \times C$. We set

$$\mathscr{D}^j(K, C) := |\mathfrak{D}^j(K, C)|\,.$$

Note that $\mathscr{D}^\infty(K, C)$ is just $\mathscr{D}(K, C)$, and that there is a forgetful map $\mathscr{D}^i(K, C) \to \mathscr{D}^j(K, C)$ whenever $i \geqslant j$.

Corollary 2·3. *Let* $\dim_{CW} K = k$, $\dim_{CW} C = c$ *and suppose that* $k, c \leqslant j$. *Then the forgetful map*

$$\mathscr{D}(K, C) \longrightarrow \mathscr{D}^j(K, C)$$

*induces a surjection on* $\pi_0$. *If furthermore* $k, c \leqslant j-1$, *then the map is also an injection on* $\pi_0$.

*Proof. Injectivity.* Suppose that two contractible homotopy pushout diagrams $A \to K \times C$ and $A' \to K \times C$ are in the same component of $\mathscr{D}^j(K, C)$. This means that there is a sequence of morphisms of $\mathfrak{D}^j(K, C)$ represented, say, by

$$A = A_0 \overset{a_0}{\longleftarrow} A_1 \overset{a_1}{\longrightarrow} \cdots \overset{a_{n-1}}{\longleftarrow} A_n = A'\,.$$

By the Mayer–Vietoris sequence, we infer that $a_i$ is a $(j-1)$-connected map of spaces. By 2·1, there exist maps $B_i \longrightarrow A_i$ for $0 \leqslant i \leqslant n$, with $B_i$ 1-connected, and where the map on reduced homology

$$H_*(B_i) \xrightarrow{\cong} H_*(K) \oplus H_*(C)$$

is an isomorphism for all $i$. It follows by [**9**] that $B_i$ is the homotopy type of a CW complex of dimension $\leqslant j-1$; this uses the hypothesis $k, c \leqslant j-1$.

Let $A_i \longrightarrow A_{i'}$ be one of the maps in the above sequence, where $|i - i'| = 1$. By elementary obstruction theory, there exists a map $B_i \longrightarrow B_{i'}$ such that the diagram

$$\begin{array}{ccc} B_i & \longrightarrow & B_{i'} \\ \big\downarrow & & \big\downarrow \\ A_i & \longrightarrow & A_{i'} \end{array}$$

is commutative up to homotopy. It follows that the map $B_i \to B_{i'}$ is $(j-1)$-connected. But the isomorphism $H_*(B_i) \cong H_*(K) \oplus H_*(C)$ together with Whitehead's theorem then implies that the map $B_i \to B_{i'}$ is actually a homotopy equivalence. Let $T_i$ denote the mapping cylinder of this homotopy equivalence.

There are then maps of spaces over $K \times C$

$$A_i^{j-1} \xrightarrow{\simeq} T_i \xleftarrow{\simeq} A_{i'}^{j-1}$$

which define a finite chain of morphisms from $A \to K \times C$ to $A' \to K \times C$ in $\mathfrak{D}(K, C)$. This establishes injectivity.

*Surjectivity.* Let $B \to K \times C$ be an object of $\mathfrak{D}^j(K, C)$, with $k, c \leqslant j$. By 2.1 there exists a $(j-1)$-connected map $A \to B$, with $A$ simply connected such that the homotopy pushout of $K \leftarrow A \to C$ is contractible. Surjectivity follows.

## 3. *Proof of Theorem A(1)*

3·1 The assignment

$$K, C \mapsto \mathscr{D}(K, C)$$

is a homotopy functor in the following sense: If $K \to K'$ and $C \to C'$ are homotopy equivalences, then there is an induced map $\mathscr{D}(K, C) \to \mathscr{D}(K', C')$ which is also a homotopy equivalence. The map is given by associating to a contractible homotopy pushout diagram $A \to K \times C$ the composite $A \to K \times C \to K' \times C'$.

Given $K$, let $G$ denote the geometric realization of the Kan loop group of the total singular complex of $K$. Thus $G$ is a topological group object in the compactly generated topology; it is a topological group model for the loop space of $K$. Therefore the classifying space $BG$ is homotopy equivalent to $K$. Similarly, let $H$ denote the realization of Kan loop group of the total singular complex of $C$. It follows that $\mathscr{D}(K, C) \simeq \mathscr{D}(BG, BH)$.

Let $A \to BG \times BH$ be an object of of $\mathfrak{D}(BG, BH)$, and let $\alpha \in \pi_0(\mathscr{D}(BG, BH))$ denote its path component. Let $\tilde{A}$ denote the pullback of the diagram

$$EG \times EH \to BG \times BH \leftarrow A\,.$$

Then $\tilde{A}$ is a space equipped with a free action of $G \times H$. Note that $\tilde{A}$ is also equipped with a basepoint. However, the basepoint is not left fixed under the action of $G \times H$.

Set

$$M_\alpha := \mathrm{Aut}_{G\times H}(\tilde{A})\,,$$

the equivariant homotopy automorphisms of $\tilde{A}$ which preserve the basepoint. This is by definition the topological monoid given by realizing (the underlying simplicial set) of the following simplicial monoid: in simplicial degree $k$ we take the $G\times H$-maps

$$\tilde{A}\wedge \Delta^k_+ \to \tilde{A}$$

whose underlying map of spaces is a weak homotopy equivalence (here $\tilde{A}\wedge\Delta^k_+$ denotes $\tilde{A}\times\Delta^k$ with $*\times\Delta^k$ collapsed to a point); by the equivariant Whitehead theorem, such a map is also a $G\times H$-homotopy equivalence in the strong sense. Therefore $M_\alpha$ has the structure of a group-like topological monoid (in the compactly generated topology).

Let $\mathscr{D}_\alpha(BG, BH)$ denote connected component of $\mathscr{D}(BG, BH)$ corresponding to $\alpha\in\pi_0(\mathscr{D}(BG, BH))$. There is then a decomposition

$$\mathscr{D}(BG, BH) = \coprod_\alpha \mathscr{D}_\alpha(BG, BH)\,.$$

To prove Theorem A(1) it will be sufficient to establish the following:

*Claim* 3·2. There is a homotopy equivalence

$$BM_\alpha \simeq \mathscr{D}_\alpha(BG, BH)\,.$$

To prove the claim, let $h\mathbb{C}(G\times H)_\alpha$ be the category whose *objects* are $G\times H$-spaces $Y$ equipped with a (non-equivariant) basepoint. Additionally, we assume that there exists a chain of equivariant weak equivalences starting with $\tilde{A}$ and terminating in $Y$, each of which preserves the basepoint. Objects are also subject to the condition that they are the weak equivariant homotopy type of a free $G\times H$-CW complex. A *morphism* of $h\mathbb{C}(G\times H)_\alpha$ is a strong equivariant homotopy equivalence which preserves the basepoint.

There is a functor $F\colon \mathscr{D}_\alpha(BG, BH)\to h\mathbb{C}(G\times H)_\alpha$ which maps an object $A'\to BG\times BH$ to the pullback of

$$EG\times EH \to BG\times BH \leftarrow A'\,.$$

There is a functor in the other direction defined by mapping a space $Y$ with $(G\times H)$-action to its Borel construction

$$Y\times_{G\times H} EG\times EH\,,$$

where the latter is a considered as a space over $BG\times BH$ by means of the projection away from $Y$.

It is relatively straightforward to check that the composites give functors

$$\mathscr{D}_\alpha(BG, BH)\to\mathscr{D}_\alpha(BG, BH) \quad\text{and}\quad h\mathbb{C}(G\times H)_\alpha\to h\mathbb{C}(G\times H)_\alpha$$

which admit natural transformations to the identity (for an argument along these lines, see [**8**, 2·1·3] and [**5**, 1·3]).

Thus $\mathscr{D}_\alpha(BG, BH)$ and $h\mathbb{C}(G\times H)_\alpha$ have homotopy equivalent realizations. We are therefore reduced to the problem of showing that the realization of $h\mathbb{C}(G\times H)_\alpha$ is equivalent to the classifying space of $M_\alpha$.

Let $h\mathbb{C}_\bullet(G \times H)_\alpha$ denote the simplicial category which in degree $k$ has the same objects as $h\mathbb{C}(G \times H)_\alpha$, but where a morphism from an object $Y$ to an object $Z$ is a $k$-parameter family of morphisms of $h\mathbb{C}(G \times H)_\alpha$, i.e. a map $Y \wedge \Delta^k_+ \to Z$ such that $Y \times t \to Z$ is a morphism of $h\mathbb{C}(G \times H)_\alpha$ for each $t \in \Delta^k$.

By considering $h\mathbb{C}(G \times H)_\alpha$ as a simplicial subcategory of $h\mathbb{C}_\bullet(G \times H)_\alpha$ using constant families, we obtain an inclusion functor

$$h\mathbb{C}(G \times H)_\alpha \to h\mathbb{C}_\bullet(G \times H)_\alpha$$

The simplicial monoid $M_\alpha = \mathrm{Aut}_{G\times H}(\tilde{A})$ is just the simplicial full subcategory of $h\mathbb{C}_\bullet(G \times H)_\alpha$ which in every degree consists of the single object $\tilde{A}$.

The claim now follows by

LEMMA 3·3. *The inclusions*

$$h\mathbb{C}(G \times H)_\alpha \to h\mathbb{C}_\bullet(G \times H)_\alpha \leftarrow M_\alpha$$

*induce equivalences on realization* (*i.e. classifying spaces*).

Waldhausen has proven an analogue of this result for categories of spaces having equivariant basepoints (see [**8**, 2·2·5]. The proof of 3·3 uses what is essentially the same argument. We therefore leave the task of checking details to the reader. This completes the proof of Theorem A(1).

3·4 We end this section with an alternative description up to homotopy of the topological monoids $M_\alpha$.

Let $A \to K \times C$ be an object of $\mathfrak{D}(K, C)$ which represents a point in the component $\mathscr{D}_\alpha(K, C)$. Without loss in generality, we may assume that $A \to K \times C$ is a fibration, and that $K = BG, C = BH$. Let $\mathrm{Aut}_{/K\times C}(A)$ denote the topological monoid of fibre homotopy equivalences $A \to A$ which preserve the basepoint. Then there is a weak homotopy equivalence

$$\mathrm{Aut}_{/K\times C}(A) \simeq M_\alpha\,,$$

which is induced by taking the base change of a fibre homotopy equivalence $A \to A$ along $EG \times EG \to K \times C$.

## 4. *Proof of Theorem A(2)*

4·1. We begin by recalling the EHP-sequence for stable homotopy, which for a space $X$, gives a functorially associated sequence

$$X \xrightarrow{E} Q(X) \xrightarrow{H} Q(X^{\wedge 2})_{h\mathbb{Z}/2}\,, \tag{4·2}$$

where $Q(X) = \Omega^\infty\Sigma^\infty(X)$ is the stable homotopy of $X$,

$$Q(X^{\wedge 2})_{h\mathbb{Z}/2} \coloneqq Q(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} X^{\wedge 2})$$

denotes the quadratic construction on $X$, the map $E$ is the evident inclusion and $H$ denotes the *Segal–Snaith–Hopf invariant* (see [**7**]). It is known that this sequence is a fibration up to homotopy in range $3\mathrm{conn}\,(X) + 1$, i.e. there is a preferred map from $X$ into the homotopy fibre of $H$ which is $(3\mathrm{conn}\,(X) + 1)$-connected.

Clearly, a map $X \to Y$ of based spaces induces a map $Q(X^{\wedge 2})_{h\mathbb{Z}/2} \to Q(Y^{\wedge 2})_{h\mathbb{Z}/2}$. More generally, the following is also true.

Lemma 4·3. *The homotopy functor*

$$X \mapsto Q(X^{\wedge 2})_{h\mathbb{Z}/2}$$

*factors through the stable category up to natural equivalence. In particular, a stable map* $\Sigma^m X \to \Sigma^m Y$ *induces a map*

$$Q(X^{\wedge 2})_{h\mathbb{Z}/2} \to Q(Y^{\wedge 2})_{h\mathbb{Z}/2}\,.$$

*Proof.* The quadratic construction can be performed in the category of spectra. Namely, to a spectrum $\mathbf{E}$ we can associate the smash product $\mathbf{E}^{\wedge 2}$; this is a spectrum with $\mathbb{Z}/2$-action. We may therefore form its homotopy orbit spectrum $\mathbf{E}^{\wedge 2}_{h\mathbb{Z}/2}$. Let $\Omega^\infty \mathbf{E}^{\wedge 2}_{h\mathbb{Z}/2}$ denote its zeroth space. If $\mathbf{E} = \Sigma^\infty X$, it follows that $\Omega^\infty \mathbf{E}^{\wedge 2}_{h\mathbb{Z}/2}$ is just $Q(X^{\wedge 2})_{h\mathbb{Z}/2}$, up to natural equivalence.

Corollary 4·4. *Let* $A \to K \times C$ *represent a contractible homotopy pushout diagram. Then there is a natural weak equivalence*

$$Q(A) \simeq Q(K \vee C)\,,$$

*and there is a natural weak equivalence*

$$\begin{aligned} Q(A^{\wedge 2})_{h\mathbb{Z}/2} &\simeq Q((K \vee C)^{\wedge 2})_{h\mathbb{Z}/2} \\ &\simeq Q(K^{\wedge 2})_{h\mathbb{Z}/2} \times Q(C^{\wedge 2})_{h\mathbb{Z}/2} \times Q(K \wedge C)\,. \end{aligned}$$

*Proof.* The composite map

$$\Sigma A \xrightarrow{c} \Sigma A \vee \Sigma A \to \Sigma K \vee \Sigma C = \Sigma(K \vee C)$$

is a weak homotopy equivalence, where $c$ denotes the comultiplication (the pinch map). Applying 4·3 to this map yields the second part.

For the first part, we just apply the functor $\Omega Q(-)$ to this map.

4·5. *Construction.* Let

$$H_K\colon Q(K) \to Q(K^{\wedge 2})_{h\mathbb{Z}/2} \quad \text{and} \quad H_C\colon Q(C) \to Q(C^{\wedge 2})_{h\mathbb{Z}/2}$$

denote the Segal–Snaith–Hopf maps. If

$$f\colon K \vee C \to Q(K \wedge C)$$

is a map, then we obtain an induced map

$$f_Q\colon Q(K \vee C) \to Q(K \wedge C)$$

by taking its infinite loop envelope. Also, let

$$\pi\colon Q(K \vee C) \to Q(K \wedge C)$$

denote the (non-infinite loop) map defined by the composition

$$Q(K \vee C) \xrightarrow{\simeq} Q(K) \times Q(C) \xrightarrow{\wedge} Q(K \wedge C)\,,$$

where the second map is defined by

$$(u\colon S^{n_1} \to S^{n_1} \wedge K, v\colon S^{n_2} \to S^{n_2} \wedge C) \mapsto (u \wedge v\colon S^{n_1} \wedge S^{n_2} \to S^{n_1} \wedge S^{n_2} \to K \wedge C)\,.$$

Lastly, let

$$f_Q + \pi\colon Q(K \vee C) \to Q(K \wedge C)$$

denote the composition

$$Q(K \vee C) \xrightarrow{f_Q,\,\pi} Q(K \wedge C) \times Q(K \wedge C) \xrightarrow{+} Q(K \wedge C)\,,$$

where the second map of the composite denotes addition.

Then $H_K, H_C$ and $f_Q$ and $\pi$ combine to yield a map

$$(H_K, H_C, f_Q + \pi)\colon Q(K \vee C) \longrightarrow Q(K^{\wedge 2})_{h\mathbb{Z}/2} \times Q(C^{\wedge 2})_{h\mathbb{Z}/2} \times Q(K \wedge C)\,.$$

Let $P_2(K)$ denote the homotopy fibre of $H_K$, $P_2(C)$ denote the homotopy fibre of $H_C$ and let $W_f$ denote the homotopy fibre of $(H_K, H_C, f_Q)$. There is then a commutative diagram

$$\begin{array}{ccccc}
W_f & \longrightarrow & Q(K \vee C) & \xrightarrow{(H_K,\, H_C,\, f_Q+\pi)} & Q(K^{\wedge 2})_{h\mathbb{Z}/2} \times Q(C^{\wedge 2})_{h\mathbb{Z}/2} \times Q(K \wedge C) \\
\downarrow & & \downarrow & & \downarrow \\
P_2(K) \times P_2(C) & \longrightarrow & Q(K) \times Q(C) & \xrightarrow[(H_K,\, H_C)]{} & Q(K^{\wedge 2})_{h\mathbb{Z}/2} \times Q(C^{\wedge 2})_{h\mathbb{Z}/2}
\end{array}$$

where the vertical maps are the evident ones and the horizontal maps are fibrations up to homotopy. Let $B_f$ be defined as the homotopy pullback of the diagram

$$K \times C \rightarrow P_2(K) \times P_2(C) \leftarrow W_f\,.$$

Then $B_f$ is a space over $K \times C$.

PROPOSITION 4·6. *The homotopy pushout of*

$$K \leftarrow B_f \rightarrow C$$

*is* $3\min(r, s)$*-connected, where* $r = \operatorname{conn} K$ *and* $s = \operatorname{conn} C$.

*Proof.* By homotopy excision, the EHP-sequence

$$K \rightarrow Q(K) \rightarrow Q(K^{\wedge 2})_{h\mathbb{Z}/2}$$

is also a cofibration sequence up to homotopy in the range $3r+2$. This means that the map from the homotopy cofibre of $K \rightarrow Q(K)$ into $Q(K^{\wedge 2})_{h\mathbb{Z}/2}$ is $(3r+2)$-connected. Similarly, $C \rightarrow Q(C) \rightarrow Q(C^{\wedge 2})_{h\mathbb{Z}/2}$ is a cofibration sequence up to homotopy in the range $3s+2$.

For any $f\colon K \vee C \rightarrow Q(K \wedge C)$ the map

$$(H_K, H_C, f_Q + \pi)\colon Q(K \vee C) \rightarrow Q(K^{\wedge 2})_{h\mathbb{Z}/2} \times Q(K^{\wedge 2})_{h\mathbb{Z}/2} \times Q(K \wedge C)$$

is $(\min(r, s)+1)$-connected. Therefore, homotopy excision implies that the homotopy fibration sequence

$$W_f \rightarrow Q(K \vee C) \xrightarrow{(H_K,\, H_C,\, f_Q+\pi)} Q(K^{\wedge 2})_{h\mathbb{Z}/2} \times Q(K^{\wedge 2})_{h\mathbb{Z}/2} \times Q(K \wedge C)$$

is also a cofibration sequence up to homotopy in the range $3\min(r, s) + 2$. As $B_f$ is defined to be the homotopy pullback of $W_f$ along the $(3\min(r, s)+1)$-connected map $K \times C \rightarrow P_2(K) \times P_2(C)$, it follows that $B_f \rightarrow W_f$ is also $(3\min(r, s)+1)$-connected.

Consequently, the sequence

$$B_f \to Q(K \vee C) \xrightarrow{(H_K,\, H_C,\, f_Q + \pi)} Q(K^{\wedge 2})_{h\mathbb{Z}/2} \times Q(C^{\wedge 2})_{h\mathbb{Z}/2} \times Q(K \wedge C)$$

is a cofibration up to homotopy in the range $3 \min(r, s) + 2$.

Consider the diagram

$$\begin{array}{ccccc} K & \longrightarrow & Q(K) & \longrightarrow & Q(K^{\wedge 2})_{h\,\mathbb{Z}/2} \\ \uparrow & & \uparrow & & \uparrow \\ B_f & \longrightarrow & Q(K \vee C) & \longrightarrow & Q(K^{\wedge 2})_{h\mathbb{Z}/2} \times Q(C^{\wedge 2})_{h\mathbb{Z}/2} \times Q(K \wedge C) \\ \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & Q(C) & \longrightarrow & Q(C^{\wedge 2})_{h\mathbb{Z}/2} \end{array}$$

The homotopy pushout of the middle column can be identified with the join $Q(K) * Q(C)$ (here we are using the identification $Q(K \vee C) \simeq Q(K) \times Q(C)$). The homotopy pushout of the third column admits a natural map to $\Sigma Q(K \wedge C)$ which is a weak equivalence in our range. The induced map of homotopy pushouts from the middle column to the third column can be identified up to homotopy within the range as the Hopf construction of the map

$$Q(K) \times Q(C) \xrightarrow{f_Q + \pi} Q(K \wedge C)\,.$$

The homotopy class of the Hopf construction is independent of the choice of $f$ since the Hopf construction is essentially a suspended difference map for the pair $(f_Q + \pi, f_Q)$. Thus we may as well take $f$ to be the constant map to the base point. In this instance the Hopf construction becomes a homotopy equivalence in our range. Therefore the Hopf construction of $f_Q + \pi$ is also a homotopy equivalence in our range.

Since the sequence

$$\operatorname{hocolim}(K \leftarrow B_f \to C) \to Q(K) * Q(C) \to \Sigma Q(K \wedge C)$$

is a cofibration up to homotopy in the range $3 \min(r, s) + 2$, it follows that $\operatorname{hocolim}(K \leftarrow B_f \to C)$ is $3 \min(r, s)$-connected, as claimed. □

COROLLARY 4·7. *The assignment* $f \mapsto B_f$ *induces a map*

$$\Phi\colon \{K \vee C, K \wedge C\} \to \pi_0(\mathscr{D}^j(K, C))\,,$$

*provided that* $j \leqslant 3 \min(r, s)$.

4·8. *Construction.* Let $A \to K \times C$ be an object of $\mathfrak{D}(K, C)$. Then the Segal–Snaith–Hopf invariant for $A$ is then

$$H_A\colon Q(A) \to Q(A^{\wedge 2})_{h\mathbb{Z}/2}\,.$$

Let $K \vee C \to Q(A)$ be the unique map up to homotopy such that the composition

$$K \vee C \to Q(A) \xrightarrow{\simeq} Q(K \vee C)$$

coincides with the canonical map up to homotopy, where the second map in this composite is given by the first equivalence of 4·4.

Using as well the second equivalence of 4·4, we may take the composite

$$K \vee C \to Q(A) \xrightarrow{H_A} Q(A^{\wedge 2})_{h\mathbb{Z}/2} \xrightarrow{\simeq} Q(K^{\wedge 2})_{h\mathbb{Z}/2} \times Q(C^{\wedge 2})_{h\mathbb{Z}/2} \times Q(K \wedge C)$$

projecting onto the third factor, we obtain a map

$$g_A\colon K \vee C \to Q(K \wedge C).$$

The assignment

$$(A \to K \times C) \mapsto g_A$$

induces a map

$$\Psi\colon \pi_0(\mathscr{D}(K,C)) \to \{K \vee C, K \wedge C\}\,.$$

*Remark* 4·9. Here is a much simpler description of $\Psi$: A contractible homotopy pushout $A \to K \times C$ induces a map $Q(A) \to Q(K \wedge C)$ by taking the composite $A \to K \times C \to K \wedge C$ and applying $Q(-)$. Let $K \vee C \to Q(A)$ be any map such that the composite $K \vee C \to Q(A) \simeq Q(K \vee C)$ coincides with the canonical map up to homotopy. Then the composite

$$K \vee C \to Q(A) \to Q(K \wedge C)$$

represents the application of $\Psi$ to the diagram. We leave it to the reader to verify this.

The following is now straightforward.

Lemma 4·10. *Let $j \leqslant 3\min(r,s)$. Then the composite*

$$\pi_0(\mathscr{D}(K,C)) \xrightarrow{\Psi} \{K \vee C, K \wedge C\} \xrightarrow{\Phi} \pi_0(\mathscr{D}^j(K,C))$$

*coincides with the forgetful map.*

Taking $j = 3\min(r,s)$ and applying 2·3, we conclude

Corollary 4·11. *The map*

$$\pi_0(\mathscr{D}(K,C)) \xrightarrow{\Psi} \{K \vee C, K \wedge C\}$$

*is injective, provided that $k, c \leqslant 3\min(r,s) - 1$.*

To complete the proof of Theorem A(2), it suffices to check that traversing

$$\{K \vee C, K \wedge C\} \xrightarrow{\Phi} \pi_0(\mathscr{D}^j(K,C)) \twoheadleftarrow \pi_0(\mathscr{D}(K,C)) \xrightarrow{\Psi} \{K \vee C, K \wedge C\}$$

from left to right is the identity. That the middle map is a surjection follows from the hypothesis $k, c \leqslant 3\min(r,s)$ and 2·3.

Suppose that $f\colon K \vee C \to Q(K \wedge C)$ is a map. Since by assumption $k, c \leqslant 3\min(r,s)$, it follows that $f$ factors uniquely up to homotopy as a map $K \vee C \to K \wedge C$. By abuse of notation, we also call the latter map $f$. Let $A_f \to K \times C$ be the result of applying the construction $\Phi$ to $f$, and then lifting arbitrarily by means of the surjection $\pi_0(\mathscr{D}(K,C)) \twoheadrightarrow \pi_0(\mathscr{D}^j(K,C))$.

Define a map $\Sigma(K \times C) \to \Sigma K \wedge C$ by taking the composite

$$\Sigma(K \times C) \xrightarrow{p_K + p_C} \Sigma K \vee \Sigma C \xrightarrow{\Sigma f} \Sigma K \wedge C\,,$$

where $p_K$ and $p_C$ are the projection maps onto $K$ and $C$ respectively. Call this map $f_\sharp$, and note that the composite of it with the inclusion $\Sigma(K \vee C) \to \Sigma(K \times C)$ coincides with $f$ up to homotopy.

It is straightforward to check, using the definition of $A_f$ via the construction 4·5, that there is a cofibration sequence up to homotopy

$$\Sigma A_f \to \Sigma(K \times C) \xrightarrow{\Sigma\pi + f_\sharp} \Sigma K \wedge C\,,$$

where the first map is the suspension of the natural map $A_f \to K \times C$ and $\pi$: $K \times C \to K \wedge C$ now denotes the quotient map $K \times C \to K \wedge C$. Moreover, the sequence splits up to homotopy. In fact, let $K * C \to \Sigma(K \times C)$ denote the Hopf construction of the identity map of $K \times C$. Then, as is well-known, the composite

$$K * C \xrightarrow{\text{Hopf}} \Sigma(K \times C) \to \Sigma K \wedge C$$

is a homotopy equivalence. Let $\Sigma K \wedge C \to K * C$ be a homotopy inverse. Then the composite $\Sigma K \wedge C \to K * C \to \Sigma(K \times C)$ gives a section to $\Sigma\pi$; denote this section by $s$.

We claim that $(\Sigma\pi + f_\sharp) \circ s$ is stably homotopic to the identity. To see this, it is sufficient to prove that $f_\sharp \circ s$ is null-homotopic. Now, $f_\sharp \circ s$ is given by the composition

$$\Sigma K \wedge C \xrightarrow{\simeq} K * C \xrightarrow{\text{Hopf}} \Sigma(K \times C) \xrightarrow{p_K + p_C} \Sigma K \vee \Sigma C \xrightarrow{\Sigma f} \Sigma K \wedge C\,,$$

and the composite in the middle $(p_K + p_C) \circ \text{Hopf} : K * C \to \Sigma K \vee \Sigma C$ coincides with the sum of Hopf construction of the map $p_K$: $K \times C \to K$ with the Hopf construction of the map $p_C$: $K \times C \to K$. Each of these maps has a null-homotopic Hopf construction (e.g. the map $\text{Hopf}(p_K)$: $K * C \to \Sigma K$ factors as $K * C \to K * \text{pt} \to \Sigma K$), so it follows that $f_\sharp \circ s$ is null-homotopic. This establishes the claim.

Now suppose that $g$: $K \vee C \to K \wedge C$ is the result of applying $\Phi$, then arbitrarily lifting from $\pi_0(\mathscr{D}^j(K, C))$ to $\pi_0(\mathscr{D}(K, C))$, and then applying $\Psi$. It follows from the definition of $A_f$ and $g$ that there is a homotopy commutative diagram of the form

$$\begin{array}{ccccc} \Sigma A_f & \longrightarrow & \Sigma(K \times C) & \xrightarrow{\Sigma\pi + f_\#} & \Sigma K \wedge C \\ \| & & \| & & \downarrow h \\ \Sigma A_f & \longrightarrow & \Sigma(K \times C) & \xrightarrow{\Sigma\pi + g_\#} & \Sigma K \wedge C \end{array}\,,$$

where $h$: $\Sigma K \wedge C \to \Sigma K \wedge C$ denotes a suitable homotopy equivalence. We therefore obtain the equation

$$h(\Sigma\pi + f_\sharp) \simeq \Sigma\pi + g_\sharp\,.$$

Applying the section $s$ on the right, we verify after a finite number of suspensions that

$$h \simeq h(\Sigma\pi + f_\sharp)s \simeq (\Sigma\pi + g_\sharp)s \simeq \text{id}\,.$$

thus $h$ is stably the identity map. From this it follows that $\Sigma\pi + f_\sharp$ is stably homotopic to $\Sigma\pi + g_\sharp$, whence $f_\sharp$ is stably homotopic to $g_\sharp$. We conclude that $f$ is homotopic to $g$. This completes the proof of Theorem A(2).

## 5. *Proof of Theorem B*

5·1. Our Theorem B will be a consequence of Quillen's Theorem B (see [**6**]). Let us recall the statement of the latter. Let $f\colon C \to D$ be a functor. For an object $d \in D$, recall that the *comma category* $d\backslash f$ has *objects* given by pairs $(c, u)$ such that $c$ is an object of $C$ and $u$ is a a morphism $d \to f(c)$. A *morphism* of $d\backslash f$ from $(c, u)$ to $(c', u')$ is a map $v\colon c \to c'$ such that $f(v)u = u'$. Note that a morphism $w\colon d \to d'$ of $D$ induces a *base change* map

$$w^*\colon d'\backslash f \to d\backslash f\,.$$

Let $f^{-1}(d)$ be the subcategory of $C$ which consists of objects $c$ such that $f(c) = d$. A morphism $c \to c'$ is a map which induces the identity map of $d$ after application of $f$. There is a functor $d = f^{-1}(d) \to d\backslash f$ defined by $c \mapsto (c, \mathrm{id})$.

Theorem 5·2 (Quillen [**6**]). *Suppose that for every $d \in D$ that the functor*

$$f^{-1}(d) \to d\backslash f$$

*induces an equivalence on realizations. Additionally, suppose that every morphism $w\colon d \to d'$ induces a homotopy equivalence*

$$w^*\colon d'\backslash f \to d\backslash f\,.$$

*Then the diagram*

$$\begin{array}{ccc} f^{-1}d & \longrightarrow & C \\ \downarrow & & \downarrow \\ * & \longrightarrow & D \end{array}$$

*is homotopy cartesian, i.e. the sequence $f^{-1}d \to C \to D$ is a fibration up to homotopy after taking realization.*

In the introduction we defined $\mathscr{E}(K, C) = |\mathfrak{E}(K, C)|$, where $\mathfrak{E}(K, C)$ is the category whose objects are contractible homotopy pushouts $A' \to K' \times C'$ with $K'$ homotopy equivalent to $K$ and $C'$ homotopy equivalent to $C$. A morphism $(A' \to K' \times C') \to (A'' \to K'' \times C'')$ is specified by homotopy equivalences $K' \to K''$, $C' \to C''$ and $A' \to A''$ such that the diagram

$$\begin{array}{ccc} A' & \longrightarrow & K' \times C' \\ \downarrow & & \downarrow \\ A'' & \longrightarrow & K'' \times C'' \end{array}$$

is commutative. Let $h_K\mathbf{Top}_*$ denote the category whose *objects* are based spaces $X$

which are homotopy equivalent to $K$; a *morphism* is a homotopy equivalence of based spaces. There is then a forgetful functor

$$\begin{aligned} \mathfrak{E}(K,C) &\xrightarrow{f} h_K\mathbf{Top}_* \times h_C\mathbf{Top}_* \\ (A' \to K' \times C') &\longmapsto K' \times C' \end{aligned}$$

If $(K',C') \in h_K\mathbf{Top}_* \times h_C\mathbf{Top}_*$ is an object, then $f^{-1}(K',C')$ is just $\mathfrak{D}(K',C')$.

Moreover, by Waldhausen [**8**, 2·2·5], the realization of $h_K\mathbf{Top}_* \times h_C\mathbf{Top}_*$ is given by $BG(K) \times BG(C)$, where $G(K)$ denotes the topological group of based homotopy automorphisms of $K$.

To establish the first part of our Theorem B, we need to show that there is a homotopy fibration

$$\mathscr{D}(K,C) \to \mathscr{E}(K,C) \to BG(K) \times BG(C)\,. \tag{5·3}$$

hence, it will be sufficient to check that $f$: $\mathfrak{E}(K,C) \to h_K\mathbf{Top}_* \times h_C\mathbf{Top}_*$ satisfies the hypotheses of Quillen's Theorem B.

For an object $d := (K',C') \in h_K\mathbf{Top}_* \times h_C\mathbf{Top}_*$, the comma category $d\backslash f$ has objects given by $(U \to X\times Y, a, b)$ where $U \to X\times Y$ denotes a contractible homotopy pushout, and $a$: $K' \to X$ and $b$: $C' \to Y$ are based homotopy equivalences. The functor $f^{-1}(d) \to d\backslash f$ in this instance is given by

$$(A' \to K' \times C') \longmapsto (A' \to K' \times C', \mathrm{id}_{K'}, \mathrm{id}_{C'})\,.$$

Let $g$: $d\backslash f \to f^{-1}(d)$ be the functor given by the assignment

$$(U \to X \times Y, a, b) \longmapsto (U^* \to K' \times C')\,,$$

where $U^*$ is defined as follows: let $U^{\mathrm{fib}} \to X \times Y$ be $U \to X \times Y$ converted into a fibration. Then $U^*$ is the pullback of this fibration along the map $(a,b)$: $K' \times C' \to X \times Y$. The composite functor $g \circ f$: $f^{-1}(d) \to f^{-1}(d)$ is given by

$$(A' \to K' \times C') \longmapsto (A'^{\mathrm{fib}} \to K' \times C')\,,$$

where $A'^{\mathrm{fib}} \to K' \times C')$ is the result of converting $A' \to K' \times C'$ into a fibration. Since there is a natural homotopy equivalence $A' \to A'^{\mathrm{fib}}$ which covers the identity map of $K' \times C'$, we obtain a natural transformation of the identity functor to $g \circ f$.

On the other hand, $f \circ g$: $d\backslash f \to d\backslash f$ is the functor which is given by

$$(U \to X \times Y, a, b) \to (U^* \to K' \times C', \mathrm{id}_{K'}, \mathrm{id}_{C'})\,,$$

where $U^*$ is the result of pulling back the fibration $U^{\mathrm{fib}} \to X \times Y$ along the map $(a,b)$. The chain of homotopy equivalences

$$U^* \xrightarrow{\simeq} U^{\mathrm{fib}} \xleftarrow{\simeq} U$$

define a chain of natural transformations from $f\circ g$ to the identity. We infer that $f$ is a homotopy equivalence. This establishes the first condition of Quillen's Theorem B for the functor $f$.

To establish the second condition, let $d := (X,Y) \to (W,Z) := d'$ denote a morphism of $h_K\mathbf{Top}_* \times h_C\mathbf{Top}_*$, i.e. a based homotopy equivalence $r_1$: $X \to W$ and a based homotopy equivalence $r_2$: $Y \to Z$. We need to show that the base-change functor $(r_1, r_2)^*$: $d'\backslash f \to d\backslash f$ is a homotopy equivalence.

Let $s_i$ denote based homotopy inverses to $r_i$, $i = 1, 2$. Then $(s_1, s_2)$ is a morphism of $h_K\mathbf{Top}_* \times h_C\mathbf{Top}_*$ and therefore induces a base-change functor $(s_1, s_2)^*\colon d'\backslash f \to d\backslash f$. Consider the composite

$$d'\backslash f \xrightarrow{(r_1,r_2)^*} d\backslash f \xrightarrow{(s_1,s_2)^*} d'\backslash f\,.$$

This is given on objects by

$$(A' \to K' \times C', a, b) \quad\longmapsto\quad (A' \to K' \times C', r_1 s_1 a, r_2 s_2 b)\,.$$

Choose homotopies $H_i$ from $r_i \circ s_i$ to the identity. The $H_i$ then determine a simplicial homotopy $(s_1, s_2)^* \circ (r_1, r_2)^*$ to the identity after taking nerves. Similarly, $(r_1, r_2)^* \circ (s_1, s_2)^*$ is simplicially homotopic to the identity on the level of nerves.

The above arguments establish the conditions for Quillen's Theorem B for the functor $f\colon \mathfrak{E}(K, C) \to h_K\mathbf{Top}_* \times h_C\mathbf{Top}_*$; this establishes that 5·3 is a fibration up to homotopy.

To complete the proof of our Theorem B we need to construct a section for $f$. Let $\sigma\colon h_K\mathbf{Top}_* \times h_C\mathbf{Top}_* \to \mathfrak{E}(K, C)$ be the functor defined by

$$(K', C') \mapsto (K' \vee C' \xrightarrow{\subset} K' \times C')\,.$$

Then $f \circ \sigma$ is the identity.

Finally, since $BG(K) \times BG(C)$ is a connected space, and $\sigma$ is a section, we have that $\pi_0(\mathscr{D}(K, C) = \pi_0(\mathscr{E}(K, C))$. The splitting

$$\mathbf{\Omega}\mathscr{E}(K, C) \simeq \mathbf{\Omega}\mathscr{D}(K, C) \times G(K) \times G(C)$$

is defined by

$$\mathbf{\Omega}\mathscr{D}(K, C) \times \mathbf{\Omega}(BG(K) \times BG(C)) \xrightarrow{i\times\Omega\sigma} \mathbf{\Omega}\mathscr{E}(K, C) \times \mathbf{\Omega}\mathscr{E}(K, C) \xrightarrow{m} \mathbf{\Omega}\mathscr{E}(K, C)\,,$$

where $i\colon \mathbf{\Omega}\mathscr{D}(K, C) \to \mathbf{\Omega}\mathscr{E}(K, C)$ is the inclusion, $m$ denotes loop multiplication, and we use the homotopy equivalence $G(K) \times G(C) \simeq \mathbf{\Omega}(BG(K) \times BG(C))$ (which exists because $G(K)$ and $G(C)$ are group-like). This completes the proof of our Theorem B. □

## 6. *Proof of Theorem C*

Let

$$\sigma_C\colon \mathscr{D}(K, C) \to \mathscr{D}^{sc}(\Sigma K, C)$$

denote the stabilization map defined in the introduction.

6·1. *Digression.* Let

$$Y \to Z \to C$$

maps of based spaces. Let $Z \to C$ be $\alpha$-connected and $Y \to Z$ be $\beta$-connected. Let $\mathrm{fib}(Y \downarrow Z)$ denote the homotopy fibre of $Y \to Z$. Let $\Sigma_C Y$ as usual denote the (based) homotopy colimit of $C \leftarrow Y \to C$, and let $\Sigma_C Y \to \Sigma_C Z$ denote the induced map. There is then a commutative of based spaces

$$\begin{array}{ccc} \mathrm{fib}(Y \downarrow Z) & \longrightarrow & \mathrm{fib}(C \downarrow C) \simeq * \\ \downarrow & & \downarrow \\ * \simeq \mathrm{fib}(C \downarrow C) & \longrightarrow & \mathrm{fib}(\Sigma_C Y \downarrow \Sigma_C Z) \end{array} \tag{6·2}$$

(where the maps fib $(C \downarrow C) \to$ fib $(\Sigma_C Y \downarrow \Sigma_C Z)$ are defined using the two inclusions $C \to \Sigma_C Y$).

LEMMA 6·3. *The diagram is* $(\min(\alpha, \beta) + \beta - 1)$*-homotopy cartesian. I.e. the weak map*

$$\mathrm{fib}\,(Y \downarrow Z) \to \Omega\,\mathrm{fib}\,(\Sigma_C Y \downarrow \Sigma_C Z)$$

*is* $(\min(\alpha, \beta) + \alpha - 1)$*-connected.*

*Proof.* This will be a consequence of Goodwillie's generalized Blakers–Massey theorem for $n$-dimensional cubes of spaces ([**2**, 2·5]; in our context, $n = 3$). Let us review the statement of the theorem. Suppose that $\mathscr{X}$ denotes a n-dimensional cube of spaces, specified as a functor $\mathscr{X}\colon \mathbf{2^n} \to \mathbf{Top}_*$, where $\mathbf{n} = \{1, 2, \ldots, n\}$, and $\mathbf{2^n}$ denotes the poset of subsets of $\mathbf{n}$. For any subset $T \subset \mathbf{n}$, let $\partial^T \mathscr{X}$ denote the sub-cube of dimension $|T|$ given by restricting to $\mathbf{2}^T$. Suppose that $\partial^T \mathscr{X}$ is $k(T)$-homotopy cocartesian; this means that the map

$$\underset{\overrightarrow{S \subsetneq T}}{\mathrm{hocolim}}\ \mathscr{X}(S) \longrightarrow \mathscr{X}(T)$$

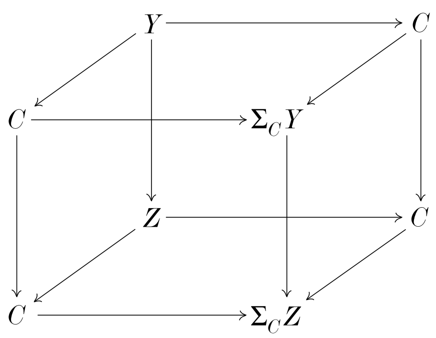
is $k(T)$-connected. Moreover, assume that $k(T) \leqslant k(U)$ whenever $T \subset U$.

Then Goodwillie's result says that $\mathscr{X}$ is $k$-homotopy cartesian, where $k$ is the minimum of

$$1 - n + \sum_{\alpha} k(T_\alpha),$$

over all partitions $\{T_\alpha\}$ of $\mathbf{n}$ by non-empty sets. The term *k-homotopy cartesian* means that the map from the initial vertex $\mathscr{X}(\varnothing)$ to the homotopy limit of the 'initially punctured' cube (i.e. delete $\mathscr{X}(\varnothing)$) has connectivity $k$.

Consider the 3-cube




Thus $Y$ is the initial vertex, $Z$ corresponds to the vertex $\{2\}$, $\Sigma_C Y$ corresponds to the vertex $\{1, 3\}$, $\Sigma_C Z$ is the terminal vertex $\{1,2,3\}$, and $C$ corresponds to the vertices $\{1\}, \{3\}\{1,2\}, \{2,3\}$. The two maps $C \to \Sigma_C Y$ are given by the natural inclusions which define $\Sigma_C Y$; similar remarks apply to $\Sigma_C Z$. As is, this cube is *not* commutative. However, it is naturally rectifiable to a commutative one by replacing the vertices labelled by $C$ with suitable mapping cylinders.

Note that the vertical homotopy fibres of the cube give rise to the square 6·2. It follows that the connectivity of the map $\mathrm{fib}\,(Y \downarrow Z) \to \Omega\,\mathrm{fib}\,(\Sigma_C Y \downarrow \Sigma_C Z)$ equals the extent to which the cube is homotopy cartesian.

From the data it is elementary to compute the numbers $k(T)$:

$$k(1) = k(3) = \min(\alpha, \beta), \quad k(2) = \alpha, \quad k(1,2) = k(2,3) = \alpha + 1,$$
$$k(1,3) = \infty, \quad k(1,2,3) = \infty\,.$$

It follows from the formula that $\mathscr{X}$ is $(\min(\alpha,\beta) + \alpha - 1)$-homotopy cartesian. □

Now suppose that $Y \to Z$ and $W \to Z$ are spaces over $Z$, where $W \to Z$ is a fibration. Let

$$F_{/Z}(Y, W)$$

denote the function space of based maps from $Y$ to $W$ which commute with the structure map to $Z$. If $W \to Z$ is not a fibration, then define $F_{/Z}(Y,W)$ as $F_{/Z}(Y, W^{\mathrm{fib}})$ where $W^{\mathrm{fib}} \to Z$ denotes $W \to Z$ converted into a fibration.

If $u\colon U \to Y$ is a cofibration and $u\colon U \to W$ is a map over $Z$, let

$$F_{/Z}(Y, W; \mathrm{rel}\, u)$$

denote the fibre of the fibration

$$F_{/Z}(Y, W) \to F_{/Z}(U, W)$$

over the composite $U \xrightarrow{u} W \to W^{\mathrm{fib}}$. Suppose additionally that $Z$ is a space over $C$. We define a suspension map

$$F_{/Z}(Y, W) \to F_{/\Sigma_C Z}(\Sigma_C Y, \Sigma_C W; \mathrm{rel}\,\phi_C)$$

where $\phi_C\colon C \vee C \to \Sigma_C W$ is the natural map. The map is defined by sending $Y \to W$ to the induced map $\Sigma_C Y \to \Sigma_C W \to (\Sigma_C W)^{\mathrm{fib}}$ considered as a morphism over $\Sigma_C Z$.

Lemma 6·4. *Suppose that* $\dim_{CW} Y = y$, *that* $W \to Z$ *is* $\alpha$*-connected and that the* $Z \to C$ *is* $\beta$*-connected. Then the map*

$$F_{/Z}(Y, W) \to F_{/\Sigma_C Z}(\Sigma_C Y, \Sigma_C W; \mathrm{rel}\,\phi_C)$$

*is* $(\min(\alpha,\beta) + \alpha - y - 1)$*-connected.*

*Proof.* For a based space $U$, let ${}^{\pm}\mathbf{Top}_*/U$ denote the following category: an *object* consists of a based map $X \to U$ together with a based map $U \vee U \to X$ such that the composite $U \vee U \to X \to U$ is the fold map. To keep notation short, we ignore the structure data and denote this object simply as $X$. A *morphism* $X \to X'$ is a map of underlying spaces which commutes with projection to $U$ and with the structure map from $U \vee U$.

If $\mathbf{Top}_*/U$ denotes the category of based spaces over the based space $Z$, it follows that suspension over $U$ defines a functor

$$\Sigma_U\colon \mathbf{Top}_*/U \to {}^{\pm}\mathbf{Top}_*/U$$

This functor admits a right adjoint,

$$\Omega_U\colon {}^{\pm}\mathbf{Top}_*/U \to \mathbf{Top}_*/U$$

which maps an object $X$ to the object given by

$$\operatorname{holim}\,(U \xrightarrow{i_-} X \xleftarrow{i_+} U),$$

where $i_\pm$ denote the restrictions of $U \vee U \to X$ to each summand; the homotopy limit is to be taken in the category of based spaces.

*Remark* 6·5. If $i_- = i_+$, then the functor $\Omega_U$ is a fibrewise loop space for retractive spaces over $U$. However, when $i_-$ and $i_+$ are different, the construction gives something else.

Using this adjoint functor pair with $U = C$, it is straightforward to verify that there is a weak equivalence

$$F_{/\Sigma_C Z}(\Sigma_C Y, \Sigma_C W; \operatorname{rel}\phi_C) \simeq F_{/\Omega_C\Sigma_C Z}(Y, \Omega_C\Sigma_C W),$$

where in the target, we are considering both $Y$ and $\Omega_C\Sigma_C W$ as spaces over $\Omega_C\Sigma_C Z$ in the evident way.

Thus, it will suffice to show that the natural map $W \to \Omega_C\Sigma_C W$ induces an $(\min(\alpha,\beta) + \alpha - y - 1)$-connected map of function spaces

$$F_{/Z}(Y, W) \to F_{/\Omega_C\Sigma_C Z}(Y, \Omega_C\Sigma_C W)\,. \tag{6·6}$$

Consider the commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & \Omega_C\Sigma_C W \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \Omega_C\Sigma_C Z \end{array} \tag{6·7}$$

The left-hand vertical homotopy fibre of (6·7) is $\operatorname{fib}(W \downarrow Z)$. The right-hand vertical homotopy fibre of (6·7) is $\Omega\operatorname{fib}(\Sigma_C W \downarrow \Sigma_C Z)$. By 6·3, the map of homotopy fibres

$$\operatorname{fib}(W \downarrow Z) \to \Omega\operatorname{fib}(\Sigma_C W \downarrow \Sigma_C Z)$$

is $(\min(\alpha,\beta)+\alpha-1)$-connected. It follows that (6·7) is $(\min(\alpha,\beta)+\alpha-1)$-homotopy cartesian.

This implies that the map (6·6) is $(\min(\alpha,\beta) + \alpha - 1 - y)$-connected, as taking function spaces drops the connectivity by the dimension of the space in the domain. This completes the proof of 6·4. □

For $Y \to Z$, let

$$\mathscr{A}_{/Z}(Y) \subset F_{/Z}(Y, Y)$$

denote the subspace given by maps $Y \to Y^{\mathrm{fib}}$ over $Z$ which are weak homotopy equivalences; thus $\mathscr{A}_{/Z}(Y)$ is just a set of connected components of $F_{/Z}(Y, Y)$. Similarly, let

$$\mathscr{A}_{/\Sigma_C Z}(\Sigma_C Y \operatorname{rel}\phi_C) \subset F_{/\Sigma_C Z}(\Sigma_C Y, \Sigma_C Y; \operatorname{rel}\phi_C)$$

be given by the maps $\Sigma_C Y \to \Sigma_C Y$ which are weak homotopy equivalences. Suspension over $C$ then induces a map

$$\mathscr{A}_{/Z}(Y) \to \mathscr{A}_{/\Sigma_C Z}(\Sigma_C Y; \mathrm{rel}\, \phi_C)\,.$$

PROPOSITION 6·8. *Suppose* $\dim_{CW} Y = y$, $Y \to Z$ *is* $\alpha$*-connected and* $Z \to C$ *is* $\beta$*-connected. Then the map*

$$\mathscr{A}_{/Z}(Y) \to \mathscr{A}_{/\Sigma_C Z}(\Sigma_C Y; \mathrm{rel}\, \phi_C)$$

*is* $(\min(\alpha, \beta) + \alpha - 1 - y)$*-connected.*

*Proof.* The diagram

$$\begin{array}{ccc}
\mathscr{A}_{/\Sigma_C Z}(Y) & \longrightarrow & \mathscr{A}_{/\Sigma_C Z}(\Sigma_C Y; \mathrm{rel}\, \phi_C) \\
\downarrow & & \downarrow \\
F_{/\Sigma}(Y, Y) & \longrightarrow & F_{/\Sigma_C Z}(\Sigma_C Y, \Sigma_C Y; \mathrm{rel}\, \phi_C)
\end{array}$$

is a set theoretic pullback (since a map $Y \to Y^{\mathrm{fib}}$ over $Z$ is a weak equivalence if and only if it is one after applying $\Sigma_C(-)$), and the vertical maps are the inclusions of a set of components. Since the bottom map is $(\min(\alpha, \beta) + \alpha - 1 - y)$-connected (by 6·4), it follows that the top one is also.

COROLLARY 6·9. *Let* $\dim_{CW} K = k$, $\dim_{CW} C = c$, *and let* $K$ *be* $r$*-connected and* $C$ *be* $s$*-connected. Suppose that* $A \to K \times C$ *denotes a contractible homotopy pushout. Then the map*

$$B\,\mathrm{Aut}_{/K\times C}(A) \to B\,\mathrm{Aut}_{/(\Sigma K)\times C}(\Sigma_C A; \mathrm{rel}\, \phi_C)$$

*is* $(2r + s - \max(k, c) + 2)$*-connected.*

*Proof.* Take $Y = A$, $Z = K \times C$ and $C = C$ in 6·8. In this instance, $\dim_{CW} Y = \max(k, c)$, $\alpha = r + s + 1$, and $\beta = r + 1$. It follows that the map

$$\mathscr{A}_{/K\times C}(A) \to \mathscr{A}_{/(\Sigma K)\times C}(\Sigma_C A; \mathrm{rel}\, \phi_C)$$

is $(2r + s - \max(k, c) + 1)$-connected.

On the other hand, $\mathrm{Aut}_{/K\times C}(A)$ was defined to be the topological monoid of based homotopy automorphisms of $A^{\mathrm{fib}} \to A^{\mathrm{fib}}$ which commute with projection to $K \times C$. The canonical map

$$\mathrm{Aut}_{/K\times C}(A) \to \mathscr{A}_{/K\times C}(A)$$

which maps a homotopy automorphism $A^{\mathrm{fib}} \to A^{\mathrm{fib}}$ to its restriction $A \to A^{\mathrm{fib}} \to A^{\mathrm{fib}}$ is therefore a weak equivalence (since $A \to A^{\mathrm{fib}}$ is a weak equivalence). Similarly, the canonical map

$$\mathrm{Aut}_{(\Sigma K)\times C}(\Sigma_C A; \mathrm{rel}\, \phi_C) \to \mathscr{A}_{(\Sigma K)\times C}(\Sigma_C A; \mathrm{rel}\, \phi_C)$$

is also a weak equivalence. It follows that the homomorphism

$$\mathrm{Aut}_{K\times C}(A) \to \mathrm{Aut}_{(\Sigma K)\times C}(\Sigma_C A; \mathrm{rel}\, \phi_C)$$

of group-like topological monoids, given by applying $\Sigma_C(-)$, is $(2r+s-\max(k,c)+1)$-connected. The map on classifying spaces is therefore $(2r+s-\max(k,c)+2)$-connected.

6·10. *Proof of Theorem C.* By 6·9 and Theorem A(1), the proof of Theorem C is reduced to the problem of establishing that

$$\sigma_C \colon \mathscr{D}(K,C) \to \mathscr{D}^{s_C}(\Sigma K, C)$$

is induces surjection on path components when $k, c \leqslant 2r+s+2$, and an injection when $k, c \leqslant 2r+s+1$.

*Surjectivity* 6·11. Let $V \to (\Sigma K) \times C$ (equipped with $C \vee C \rightarrowtail V$) be an object of $\mathfrak{D}^{\sigma_C}(\Sigma K, C)$. Note that $\boldsymbol{\Omega}_C \Sigma_C(\Sigma K \times C) \simeq (\Omega\Sigma K) \times C$. Consider the homotopy cartesian square

$$\begin{array}{ccc} B & \longrightarrow & \boldsymbol{\Omega}_C V \\ \downarrow & & \downarrow \\ K \times C & \longrightarrow & (\Omega\Sigma K) \times C \end{array} \qquad (6·12)$$

where $B$ is defined to be the pullback of the evident maps $K \times C \to (\Omega\Sigma K) \times C$ and $(\boldsymbol{\Omega}_C V)^{\mathrm{fib}} \to (\Omega\Sigma K) \times C$.

*Claim* 6·13. The homotopy pushout of $K \leftarrow B \to C$ is $(2r+s+2)$-connected. Moreover, $B$ is 1-connected.

*Proof of Claim.* Since $K \times C \to (\Omega\Sigma K) \times C$ is $2r+1 \geqslant 3$-connected, it follows that $B$ is 1-connected if and only if $\boldsymbol{\Omega}_C V$ is 1-connected. But the structure maps $i_\pm \colon C \to V$ have homotopy cofibre identified with $\Sigma K$ up to homotopy. Since $V$ and $C$ are 1-connected and since $\Sigma K$ is $r+1 \geqslant 2$ connected, Whitehead's theorem implies that $i_\pm$ are 2-connected. From this it follows that $\boldsymbol{\Omega}_C \to C$ is 2-connected as well. As $C$ is 1-connected, $\boldsymbol{\Omega}_C V$ must also be 1-connected. This proves the second part of the claim.

The first part of the claim will follow from Goodwillie's dual to the generalized Blakers–Massey theorem [**2**, 2·6]. We recall the statement of this result. Let $\mathscr{X}$ denote an $n$-cube of spaces, with the notation as in the proof of 6·3. For $T \subset \mathbf{n}$, define a subcube $\partial_T \mathscr{X}$ to be the $|\mathbf{n} - T|$-cube given by composing $\mathscr{X} \colon 2^{\mathbf{n}} \to \mathbf{Top}_*$ with the embedding $2^{\mathbf{n}-T} \to 2^{\mathbf{n}}$ defined by

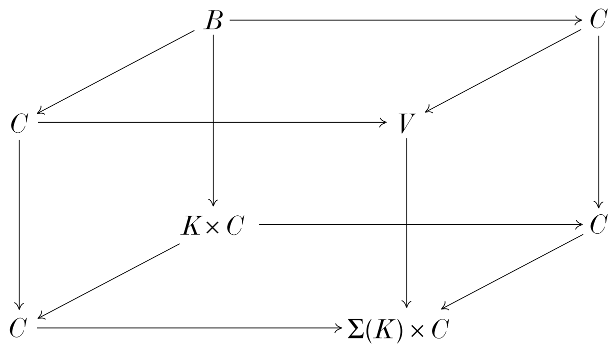
$$V \mapsto V \cup T\,.$$

Suppose for each $T \neq \mathbf{n}$ that $\partial_T \mathscr{X}$ is $k(\mathbf{n}-T)$-homotopy cartesian, and that $k(T) \leqslant k(U)$ whenever $T \subset U$. Then $\mathscr{X}$ is $k$-homotopy cocartesian, where $k$ is the minimum of

$$n - 1 + \sum_\alpha k(T_\alpha),$$

over all partitions $\{T_\alpha\}$ of $\mathbf{n}$ by non-empty subsets.

Consider the 3-cube




In our situation $n = 3$, and the numbers $k(T)$ are given by

$$k(1) = k(3) = r + 1, \quad k(2) = r + s + 1, \quad k(1,2) = k(2,3) = r + s + 1,$$
$$k(1,3) = 2r + 1, \quad k(1,2,3) = \infty\,.$$

It follows that the cube is $(2r + s + 4)$-homotopy cocartesian. This implies that the map $\Sigma_C B \to V$ is $(2r+s+3)$-connected (since the bottom face of the cube is homotopy cocartesian).

We infer that the homotopy pushout of

$$\Sigma K \leftarrow \Sigma_C B \to C$$

is $(2r + s + 3)$-connected. Consider the commutative diagram up to homotopy

$$\begin{array}{ccccc}
C \vee C & \xrightarrow{\text{fold}} & C & \longrightarrow & \Sigma C \\
\| & & \uparrow & & \uparrow \\
C \vee C & \xrightarrow{\subset} & \Sigma_C B & \longrightarrow & \Sigma B \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \Sigma K & = & \Sigma K
\end{array}$$

The horizontal rows are cofibration sequences up to homotopy. Taking homotopy pushouts vertically results therefore in a cofibration sequence up to homotopy

$$* \longrightarrow \operatorname{hocolim}(\Sigma K \leftarrow \Sigma_C B \to C) \longrightarrow \Sigma \operatorname{hocolim}(K \leftarrow B \to C)\,.$$

It follows that $\operatorname{hocolim}(K \leftarrow B \to C)$ is $(2r + s + 2)$-connected. This establishes the claim.

Using the claim, we see that the association

$$(V \to (\Sigma K) \times C, C \vee C \rightarrowtail V) \longmapsto (B \to K \times C)$$

described above gives rise to a well defined map

$$\pi_0(\mathscr{D}^{sc}(\Sigma K, C)) \to \pi_0(\mathscr{D}^{j}(K, C))$$

for $j = 2r + s + 2$, where $\mathscr{D}^j(K, C)$ is the space defined in 2·2. It is straightforward to check that traversing

$$\pi_0(\mathscr{D}^{s_C}(\Sigma K, C)) \rightarrow \pi_0(\mathscr{D}^j(K, C)) \twoheadleftarrow \pi_0(\mathscr{D}(K, C)) \xrightarrow{\sigma_C} \pi_0(\mathscr{D}^{s_C}(\Sigma K, C))$$

coincides with the identity (here we are using 2·3 to conclude that the middle map is a surjection). Surjectivity follows.

*Injectivity* 6·14. Suppose that $k, c \leqslant 2r + s + 1$. Let $j = 2r + s + 2$, and let

$$\pi_0(\mathscr{D}^{s_C}(\Sigma K, C)) \rightarrow \pi_0(\mathscr{D}^j(K, C))$$

be the map defined in the previous paragraph. By construction, the composite

$$\pi_0(\mathscr{D}(K, C)) \xrightarrow{\sigma_C} \pi_0(\mathscr{D}^{s_C}(\Sigma K, C)) \rightarrow \pi_0(\mathscr{D}^j(K, C))$$

coincides with the map induced by the inclusion $\mathscr{D}(K, C) \subset \mathscr{D}^j(K, C)$. By 2·3, the inclusion induces an injection on path components, and hence yields the injectivity of $\sigma_C$ on the level of path components. This concludes the proof of Theorem C.

## REFERENCES


[**1**] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. LNM 304 (Springer-Verlag, 1972).

[**2**] T. Goodwillie. *Calculus II, Analytic functors. K-theory* **5** (1992), 295–332.

[**3**] P. Hirschhorn. *Localization, cellularization, and homotopy colimits*, preprint.

[**4**] J. Klein. *On the homotopy embeddability of complexes in euclidean space: I. The weak thickening theorem. Math. Zeit.* **292** (1993), 145–161.

[**5**] J. Klein and J. Rognes. *A chain rule in the calculus of homotopy functors*, preprint.

[**6**] D. Quillen. *Higher algebraic K-theory I: higher K-theories; In Algebraic K-theory* I, LNM 341 (Springer-Verlag,1973), pp. 85–147.

[**7**] V. P. Snaith. *A stable decomposition of* $\Omega^n\Sigma^n X$. *J. London Math. Soc.* (2) **577** (1974), 577–583.

[**8**] F. Waldhausen. *Algebraic K-theory of spaces; In Algebraic and geometric topology*, Proceedings Rutgers 1983, LNM 1126 (Springer-Verlag, 1985), pp. 318–419.

[**9**] C. T. C. Wall. *Finiteness conditions on CW complexes; Ann. Math.* **81**, (1965), 56–69.