Math. Ann. 289, 683–701 (1991)

Mathematische Annalen

Inequivalent fibred knots whose homotopy Seifert pairings are isometric

John R. Klein[1] **and Alexander I. Suciu**[2,*]

[1] Mathematik V der Gesamthochschule, W-5900 Siegen, Federal Republic of Germany
[2] Department of Mathematics, Northeastern University, Boston, MA 02115, USA

Received August 9, 1990

0 Introduction

The problem of classification in terms of a finite list of invariants is a fundamental question of knot theory. In 1970, Levine [L] gave an algebraic classification of n-dimensional knots in S^{n+2} bounding r-connected Seifert surfaces, where $2r+1=n$, $n \geqq 3$. Levine showed that the only invariant which determines the isotopy type of such knots is the homology Seifert pairing considered up to S-equivalence. In the mid 1970's, Kearton [K] and Trotter [Tr 1; Tr 2] obtained the same classification in terms of the Blanchfield pairing. In 1980, Farber [F] extended this classification in homotopy theoretic terms to n-knots that bound r-connected Seifert surfaces, where $3r \geqq n+1$.

Suppose $V^{n+1} \subsetneqq S^{n+2}$ is a Seifert surface for an n-knot. Farber introduces a map $\Theta: V \wedge V \to S^{n+1}$, called the homotopy Seifert pairing of V, which induces the usual Seifert pairing on homology. He shows that the isometry class of Θ determines the isotopy class of V as a submanifold of S^{n+2} in a certain range of dimensions and connectivities:

0.1 **Theorem** (Farber). *Let V_1 and V_2 be two compact oriented $(n+1)$-dimensional submanifolds of the sphere S^{n+2} with ∂V_1 and ∂V_2 homotopy spheres. Let $\Theta_i: V_i \wedge V_i \to S^{n+1}$, $i=1,2$, be the corresponding homotopy Seifert pairings. Suppose that the manifolds V_1 and V_2 are r-connected, where $3r \geqq n+1$, $n \geqq 5$. If there is a homotopy equivalence $f: V_1 \xrightarrow{\sim} V_2$ for which $\Theta_2 \circ (f \wedge f)$ is homotopic to Θ_1, then there exists an isotopy of the sphere S^{n+2} transferring V_1 on V_2 with a preservation of orientations.*

Farber then treats the situation in which non-isotopic Seifert surfaces bound isotopic knots [F, Theorem 2.4], so a single knot may have several pairings associated to it. However, if the knots are assumed to be fibred, they have canonical Seifert surfaces up to isotopy, and consequently, canonical homotopy

* Partially supported by a Northeastern University Junior Research Fellowship

Seifert pairings. Hence, Theorem 0.1 gives a complete classification of fibred n-knots with r-connected fibres in the range $3r \geqq n+1$.

W. Richter (1982, unpublished; see also [Kl]) established that 0.1 is still true in one dimension better: the "stable" range $3r \geqq n$. The natural question then was: Do there exist inequivalent fibred knots having the property that their homotopy Seifert pairings are isometric?

In this paper we answer this question in the affirmative, thus showing that the statement of Farber's theorem is false outside of the range $3r \geqq n$. Our main result is

8.7 **Theorem.** *For each integer $n \geqq 9$, there exist infinitely many pairs $(L_i, \bar{L}_i)$ of fibred n-knots such that*

(1) *L_i and $\bar{L}_i$ have isometric homotopy Seifert pairings;*

(2) *the exterior of L_i is homotopically distinct from the exterior of L_j for $i \neq j$, and also from the exterior of $\bar{L}_j$ for all j;*

(3) *the knots L_i and $\bar{L}_i$ have 1-connected fibres if $n \geqq 9$, 2-connected fibres if $n \geqq 22$, and 5-connected fibres if $n \geqq 25$.*

Our examples reside in the range $r/n \leqq 1/5$, two ranges away from the stable range $1/3 \leqq r/n < 1/2$. Therefore it is still an open question as to what happens in the metastable ranges $1/5 < r/n < 1/3$.

The construction we use to provide these examples is a generalization of the frame-spinning construction of Roseman [R], as elaborated upon by the second author in [Su]. We call this new construction *diff-spinning*. The distinction between the two constructions is that we twist the framed submanifold by a self-diffeomorphism as we spin.

Diff-spinning takes fibred knots to fibred knots. The diff-spun fibre is the same as the frame-spun fibre, but the diff-spun monodromy is a twisted version of the frame-spun monodromy. If the diffeomorphism is chosen carefully, the homotopy Seifert pairing remains the same, but the homotopy type of the knot exterior changes.

We now turn to the organization of this paper. Section 1 starts with the basic definitions used throughout the paper. In Sect. 2 we establish the results we need about fibred knots. As a starting point for our constructions, we exhibit simple fibred $(2m-1)$-knots having monodromy of finite homotopy order. In Sect. 3 we define the homotopy Seifert pairing and show that for a fibred knot the isometry class of the pairing is determined by the stable homotopy class of its monodromy rel boundary. In Sect. 4 we outline the frame-spinning construction and identify its effect on the monodromy of a fibred knot. As an application, we give examples of fibred 11-knots having isometric homology Seifert pairings but non-isometric homotopy Seifert pairings. In Sect. 5 we introduce diff-spinning and establish its basic properties. In Sect. 6 we compare frame-spinning with diff-spinning and show that, under certain conditions, the homotopy Seifert pairings are the same (6.3), but the exteriors are homotopically distinct (6.5). In Sect. 7 we construct self-diffeomorphisms of $S^p \times S^q$ whose m-fold suspensions have finite homotopy order, yet which are stably homotopic to the identity. In Sect. 8 we prove our main theorem by diff-spinning the knots of Sect. 2 with the diffeomorphisms of Sect. 7, where the diffeomorphisms are chosen so that their m-fold suspensions have homotopy orders coprime to the homotopy orders of the knot monodromies.

1 Definitions and notation

1.1. We work in the category of spaces which have the homotopy type of *CW* complexes. All manifolds contained herein are assumed to be compact, orientable and smooth. Diffeomorphisms are denoted by $\cong$, homotopy equivalences by $\simeq$, embeddings by $\subseteq$, the connect sum by $\#$, and the boundary connect sum by $\natural$. S^n is the n-sphere, D^n is the n-disk, $\mathbb{R}$ is the real line, and I is the unit interval $[0,1]$. D^n_+ (resp. D^n_-) is the northern (resp. southern) hemisphere of S^n. The symbol $*$ denotes the basepoint.

For pointed spaces X and Y, the *wedge* $X \vee Y$ of X and Y is the subspace of $X \times Y$ whose points are of the form $(x,*)$ and $(*,y)$ for all $x \in X$ and $y \in Y$. Set $X_+ = X \vee S^0$. The *smash product* $X \wedge Y$ of X with Y is the quotient space of $X \times Y$ by $X \vee Y$. The *join* $X * Y$ of X and Y is the topological space obtained from the union of X, Y, and $X \times Y \times I$ by identifying $(x,y,0)$ with x and $(x,y,1)$ with y, for $x \in X$ and $y \in Y$. Taking the join of a space X with the $(r-1)$-sphere yields the space $\Sigma^r X = X * S^{r-1}$, called the *r-fold suspension* of X (alternatively, $\Sigma^r X = X \wedge S^r$). Two spaces X and Y are said to be stably homotopy equivalent, denoted $X \simeq_s Y$, if $\Sigma^r X \simeq \Sigma^r Y$ for some non-negative integer r. We let $\{X, Y\}$ denote the group of stable homotopy classes of maps from X to Y.

A self-map $f: X \to X$ is said to have *homotopy order d*, if the d-fold composition of f with itself, f^d, is homotopic to the identity, and moreover, d is the smallest such integer. We will write maps from spheres to spheres as $f_b: S^a \to S^b$, emphasizing the target dimension. The r-fold suspension of f_b will then be denoted by f_{b+r}.

If $f: X \times Y \to Z$ is a map of topological spaces, then the *Hopf construction*

$$H(f): X * Y \to \Sigma Z$$

is defined by sending (x,y,s) to the point $(f(x,y), 2s)$ if $s \leqq 1/2$, and to the point $(f(x,y), 1-2s)$ if $s > 1/2$ (cf. [Wh]). Note that $H(f) = \Sigma f \circ H(\mathrm{id}_{X \times Y})$. If $f: X \times Y \to Z$ factors through $* \times Y$, then $H(f)$ factors through $\{*\} * Y$, which is contractible, and thus $H(f)$ is null homotopic. If $\pi: X \times Y \to X \wedge Y$ is the quotient map then $H(\pi): X * Y \to \Sigma X \wedge Y$ is a homotopy equivalence. In particular, we have natural equivalences $S^{p+q+1} \simeq S^p * S^q \simeq \Sigma S^p \wedge S^q$.

1.2. We say that a submanifold $M^k \subseteq S^{n+k}$ is *framed* if its unit normal bundle $\nu_M \supseteq S^{n+k}$ is provided with a trivialization $\phi: M \times D^n \to \nu_M$. We shall always assume that M is closed, i.e., compact, connected and without boundary. Using the trivialization, we may identify a tubular neighborhood of M with $M \times D^n$. The *Thom-Pontryagin collapse* of (M, ϕ) is the map $u: S^{n+k} \to M_+ \wedge S^n$ given by the composite

$$S^{n+k} \to S^{n+k}/(S^{n+k} - \mathrm{int}(M \times D^n)) = M \times D^n / M \times S^{n-1} = M_+ \wedge S^n.$$

2 Fibred knots

2.1. An *n-knot* is an embedding $K: S^n \subseteq S^{n+2}$ (we often abuse terminology by identifying the embedding with its image). Two n-knots K and K' are *equivalent* $(K \cong K')$ if there is a diffeomorphism of S^{n+2} taking the image of K to the image of K'.

Each knot K has a tubular neighborhood of the form $S^n \times D^2 \subsetneq S^{n+2}$. The *exterior* of K is

$$X(K) = S^{n+2} - S^n \times \operatorname{int}(D^2).$$

Thus $X(K)$ is an $(n+2)$-manifold with $\partial X(K) = S^n \times S^1$. Equivalent knots have diffeomorphic exteriors. The *meridian* of K is the circle $* \times S^1$. The inclusion of the meridian in the exterior is a homology equivalence. We let $\tilde{X}(K)$ denote the universal abelian cover of $X(K)$.

By a *Seifert surface* V^{n+1} for an n-knot K, we mean a codimension one oriented submanifold of S^{n+2} whose boundary is the image of K. If K has a Seifert surface V^{n+1} which is r-connected, then K is said to be *r-simple*. If $n = 2r+1$, or if $n = 2r+2$, then K is said to be *simple*. If K is a non-trivial simple n-knot, then V is homotopy equivalent to a wedge of $(r+1)$-spheres if $n = 2r+1$, or to a complex with cells in dimensions $r+1$ and $r+2$ if $n = 2r+2$.

An n-knot K is said to be *fibred* if there is a smooth fibration $p: X(K) \to S^1$ whose restriction to the boundary is the second coordinate projection $\mathrm{pr}_2: S^n \times S^1 \to S^1$. If K is fibred then the *fibre* $V = p^{-1}(*)$ is a canonical choice of Seifert surface for the knot K (where we abuse notation and identify the knot K with its pushoff into the boundary of its tubular neighborhood).

Using the clutching construction, a fibred knot determines a *monodromy* map $\theta: V \to V$ of its canonical Seifert surface. This is a diffeomorphism such that

(m_1) the restriction of θ to $S^n = \partial V$ is isotopic to the identity;
(m_2) the map $(\mathrm{id} - \theta_*): \tilde{H}_*(V) \to \tilde{H}_*(V)$ is an isomorphism;
(m_3) the set $\{x \cdot \theta_*^{-1}(x) \mid x \in \pi_1(V)\}$ normally generates $\pi_1(V)$.

[Note that condition (m_3) is trivially satisfied if V is 1-connected.]

On the other hand, if $\theta: V \to V$ is a diffeomorphism satisfying the above conditions, then θ and a choice of isotopy $h: S^n \times I \to S^n \times I$ from $\theta_{|S^n}$ to the identity determines a fibred knot $K_\theta: S^n \subsetneq S^{n+2}$. The construction of K_θ is as follows:

Let $X_\theta = V \times_\theta S^1$ be the mapping torus of θ. Then $\partial X_\theta = S^n \times_\theta S^1$ is the boundary of the manifold $T = (S^n \times I) \times_h S^1 \cup S^n \times D^2$ $(\cong S^n \times D^2)$ where the union is taken over

$$(S^n \times 1) \times_{h_1} S^1 = S^n \times S^1 \subsetneq S^n \times D^2.$$

Let $\Sigma^{n+2} = X_\theta \cup_{\partial X_\theta} T$. A Van-Kampen and Mayer-Vietoris calculation shows that Σ^{n+2} is a homotopy $n+2$-sphere. Moreover, the core of T provides an embedding $k_\theta: S^n \subsetneq \Sigma^{n+2}$. The desired knot $K_\theta: S^n \subsetneq S^{n+2}$ is then defined to be the knot obtained from k_θ by changing if necessary the smooth structure of Σ^{n+2} on an embedded ball in the exterior of $k_\theta(S^n)$.

2.2. *A necessary condition for two fibred knots to have homotopy equivalent exteriors.* Suppose $K_i: S^n \subsetneq S^{n+2}$ are fibred knots, $i = 1, 2$, whose associated monodromies are $\theta_i: V_i \to V_i$. We say that θ_1 is *homotopically conjugate* to θ_2 if there exists a homotopy equivalence $f: V_1 \to V_2$ such that $f \circ \theta_1 \circ f^{-1} \simeq \theta_2$, where f^{-1} is a homotopy inverse of f.

2.3 **Lemma.** *If K_1 and K_2 have homotopy equivalent exteriors, then θ_1 is homotopically conjugate to θ_2.*

Proof. The exterior of K_i is the space $V_i \times_{\theta_i} S^1$. Let $g: V_1 \times_{\theta_1} S^1 \to V_2 \times_{\theta_2} S^1$ be a homotopy equivalence. Then g lifts to an equivariant map of universal abelian covers $\tilde{g}: V_1 \times \mathbb{R} \to V_2 \times \mathbb{R}$. Let $f: V_1 \to V_2$ be the composition $\mathrm{pr}_1 \circ \tilde{g} \circ i_0$, where

$\mathrm{pr}_1: V_2 \times \mathbb{R} \to V_2$ is the first coordinate projection and $i_0: V_1 \to V_1 \times \mathbb{R}$ is the inclusion of $V_1 \times 0$. Then f is the desired homotopy conjugation from θ_1 to θ_2. □

2.4. *Construction of odd dimensional simple fibred knots with monodromy of finite homotopy order.* Let a and b be coprime integers greater than one. Let $K_{a,b}: S^1 \subsetneq S^3$ be the (a, b) torus-knot. Then $K_{a,b}$ is a fibred knot with fibre homotopy equivalent to a μ-fold wedge of circles, where $\mu = (a-1)(b-1)$. With respect to this identification, the monodromy $\tau_{a,b}: \vee^\mu S^1 \to \vee^\mu S^1$ has homotopy order ab [H].

Let $m > 1$ be an integer. Consider the homotopy equivalence

$$\Sigma^{m-1}\tau_{a,b}: \vee^\mu S^m \to \vee^\mu S^m.$$

By the proof of Lemma 17 in Wall [W], there is a thickening of this map to a diffeomorphism

$$T_{a,b}: \natural^\mu S^m \times D^{m+1} \to \natural^\mu S^m \times D^{m+1}.$$

Note that $T_{a,b}$ induces the same homomorphism on H_m as $\tau_{a,b}$ on H_1. Restricting $T_{a,b}$ to the boundary we have a diffeomorphism

$$\theta_{a,b}: \#^\mu S^m \times S^m \to \#^\mu S^m \times S^m,$$

which by isotoping if necessary, may be assumed to preserve an embedded disk D^{2m}. Then the restriction of $\theta_{a,b}$ to $V_{a,b} = \#^\mu S^m \times S^m - D^{2m}$ satisfies conditions (m_1) and (m_3) trivially, and (m_2) by Lefschetz duality. Consequently, we obtain

2.5 **Proposition.** *For each integer $m \geq 1$, there exist infinitely many simple fibred $(2m-1)$-knots whose monodromies have distinct finite orders. Moreover, the exteriors of these knots are pairwise homotopy inequivalent.*

Proof. Choose pairs (a, b) whose products ab are all distinct. Then the simple fibred knots $K_{a,b}$ associated to the monodromy $\theta_{a,b}$ by 2.1 are the desired knots. By 2.3, $X(K_{a,b})$ is not homotopy equivalent to $X(K_{a',b'})$ for $(a', b') \neq (a, b)$. □

3 The homotopy Seifert pairing and its dual

3.1. *S-duality.* Suppose that $K \subsetneq S^n$ is a finite subcomplex whose dimension is less than n. If $C \subsetneq S^n$ is the exterior of a regular neighborhood N of K, then one may naturally associate a Spanier-Whitehead duality map

$$d: K * C \to S^n$$

as follows: Fix a basepoint $*$ in $S^n - (K \amalg C)$ and think of $\mathbb{R}^n$ as being $S^n - *$ via stereographic projection. We will henceforth consider K and C as being embedded in this $\mathbb{R}^n$. Let $\delta: K \times C \to S^{n-1}$ be given by the rule $\delta(k, c) = (k - c)/\|k - c\|$. The map $d: K * C \to S^n$ is then defined to be the Hopf construction of the map δ.

It is well-known that the map d induces the Alexander duality between K and C upon taking homology in dimension n and utilizing the Künneth splitting of $K * C \simeq \Sigma K \wedge C$. What seems to be less well known is that there is also a natural duality map going in the other direction,

$$d^*: S^n \to C * K,$$

defined as follows: Let N be the regular neighborhood of K given above and choose a retraction $r: N \to K$. The orientation of N induces an orientation of the normal bundle of ∂N. The Thom-Pontryagin construction then yields a degree one map (the *normal invariant* of ∂N),

$$u: S^n \to \Sigma \partial N.$$

Let $j: \partial N \to C \wedge K$ be the map induced by smashing the inclusion $i_C: \partial N \subsetneq C$ with $r_{|\partial N}: \partial N \subsetneq K$, i.e., $j(n) = i_C(n) \wedge r(n)$. Suspending j and composing with u gives a map $(\Sigma j) \circ u: S^n \to \Sigma C \wedge K$. The identification $C * K \simeq \Sigma C \wedge K$ of Sect. 1 then defines the desired duality map $d^*: S^n \to C * K$.

For any spaces X and Y, let

$$U_d(X, Y): \{X, C * Y\} \cong \{K * X, S^n * Y\},$$

and

$$V_{d^*}(X, Y): \{Y * C, X\} \cong \{Y * S^n, X * K\}$$

be the S-duality isomorphisms defined by d and d^*, respectively. These are given by the formulas

$$U_d(X, Y)(f) = (d * \mathrm{id}_Y) \circ (\mathrm{id}_K * f) \quad \text{and} \quad V_{d^*}(X, Y)(f) = (f * \mathrm{id}_K) \circ (\mathrm{id}_Y * d^*).$$

3.2 **Proposition.** $U_d(S^n, K)(d^*) = \pm \mathrm{id}_{S^n * K}$ and $V_{d^*}(S^n, K)(d) = \pm \mathrm{id}_{S^n * K}$, i.e., d and d^* are S-duals of one another up to sign.

Proof. (Sketch). By construction the proposition is true when K is a sphere or disk. As the S-duality isomorphism can be shown to commute with amalgamated unions, the proposition follows by induction on the number of cells of K (see e.g. [S]). □

3.3. *The homotopy Seifert pairing and its dual.* Suppose $V^{n+1} \subsetneq S^{n+2}$ is a Seifert surface for a knot $S^n \subsetneq S^{n+2}$. Identify a tubular neighborhood of V with $V \times [-1, 1]$ in such a way that the orientation on V together with the usual orientation of the interval agree with the standard orientation of the sphere S^{n+2}. Let $C = S^{n+2} - V \times (-1, 1)$ be the exterior of V. Then the constructions of 3.1 provide Spanier-Whitehead duality maps,

$$d: V * C \to S^{n+2}, \quad \text{and} \quad d^*: S^{n+2} \to C * V.$$

Define maps $p_\pm: V \to C$ by the formulas $p_\pm(v) = (v, \pm 1) \in V \times \pm 1 \subsetneq C$ (p_+ and p_- are the maps which push V along its outward unit normal frames). Let

$$\Theta: V * V \to S^{n+2} \quad \text{and} \quad \Psi: S^{n+2} \to C * C$$

be the maps obtained by precomposing d with the map $\mathrm{id}_V * p_+: V * V \to V * C$, and by composing the map $\mathrm{id}_C * p_-: C * V \to C * C$ with d^*.

The map $\Theta: V * V \to S^{n+2}$ was defined by Farber [F] and is called the *homotopy Seifert pairing* of the Seifert surface V.* By analogy, we call $\Psi: S^{n+2} \to C * C$ the *dual homotopy Seifert pairing* of the Seifert surface V. The justification of this terminology is provided by the following:

3.4 **Lemma.** *The map $\Psi: S^{n+2} \to C * C$ is the Spanier-Whitehead dual of the map $\Theta: V * V \to S^{n+2}$ up to sign.*

* Actually, Farber's pairing is a map $\Theta: V \wedge V \to S^{n+1}$. Our map is obtained from Farber's by suspending and using the natural equivalence $V * V \simeq \Sigma V \wedge V$ of Sect. 1

Proof. By 3.2, d^* is the Spanier-Whitehead dual of d up to sign. By duality it is sufficient to prove that $\mathrm{id}_C * p_- : C * V \to C * C$ is the Spanier-Whitehead dual of $\mathrm{id}_V * p_+ : V * V \to V * C$. But this will be true if $p_- : V \to C$ is the dual of $p_+ : V \to C$.

To see this, let $C^\pm = C \cup_{p_\pm \times 1} V \times [0,1]$ be the mapping cylinder of $p_\pm$ [$C^\pm$ is the collar of C obtained by attaching onto C the $(\pm)$-half of the tubular neighborhood of V in S^{n+2}]. Note that $C^\pm$ naturally embeds in S^{n+2}, extending the inclusion of C. Moreover, the composition

$$V \xrightarrow{p_\pm} C \subseteq C^\pm$$

is isotopic to the inclusion of V into $C^\pm$ as $V \times 0$. Note that the exterior of $p_\pm(V)$ in S^{n+2} is $C^\mp$, and the exterior of $C^\pm$ in S^{n+2} is $V \times 0$. Furthermore, the inclusion of exteriors, $V \times 0 \subseteq C^\mp$, is identical to the map

$$V \xrightarrow{p_\mp} C \subseteq C^\mp,$$

Consequently, the Spanier-Whitehead dual of $p_+ : V \to C$ is homotopic to $p_- : V \to C$. □

Suppose $\Theta_X : X * X \to S^j$ and $\Theta_Y : Y * Y \to S^j$ are arbitrary maps. Then Θ_X is said to be *isometric* (resp. *s-isometric*) to Θ_Y if there is an equivalence (resp. stable homotopy equivalence) $f : X \to Y$ such that $\Theta_Y \circ (f * f)$ is homotopic (resp. stably homotopic) to Θ_X. There is also the corresponding notion of isometry for maps $\Psi_X : S^j \to X * X$ and $\Psi_Y : S^j \to Y * Y$ which go in the opposite direction.

3.5 **Proposition.** *Let V and W be Seifert surfaces in the sphere S^{n+2} with associated homotopy Seifert pairings $\Theta_V : V * V \to S^{n+2}$, $\Theta_W : W * W \to S^{n+2}$, and dual homotopy Seifert pairings $\Psi_V : S^{n+2} \to C_V * C_V$, $\Psi_W : S^{n+2} \to C_W * C_W$. Then Θ_V and Θ_W are s-isometric if and only if Ψ_V and Ψ_W are s-isometric.*

Proof. Let $f : V \to W$ be an s-isometry from Θ_V to Θ_W. Then the S-dual of f, $f^* : C_W \to C_V$, provides an s-isometry from Ψ_V to Ψ_W by 3.4. □

3.6. *Monodromy and s-isometry.* Suppose $K : S^n \subseteq S^{n+2}$ is a fibred knot whose associated monodromy is $\theta : V \to V$. Let C be the exterior of V in S^{n+2}. Then $p_\pm : V \to C$ are homotopy equivalences. Moreover, $\theta \simeq p_+^{-1} \circ p_-$. Using the identification p_+ of V with C, we may think of the dual homotopy Seifert pairing as a map $\Psi : S^{n+2} \to V * V$.

There is another description of Ψ given as follows: Let $\partial(V \times I) \subseteq S^{n+2}$ be the boundary of a tubular neighborhood of V. Define a map $b : \partial(V \times I) \to V \wedge V$ by the composite

$$\partial(V \times I) \subseteq V \times I \xrightarrow{\mathrm{pr}_1} V \xrightarrow{\Delta_V} V \times V \xrightarrow{\mathrm{id}_V \times \theta} V \times V \xrightarrow{\pi} V \wedge V.$$

Then $\Psi : S^{n+2} \to V * V$ is given by the composite of $\Sigma b : \Sigma \partial(V \times I) \to \Sigma V \wedge V$ with the normal invariant $u : S^{n+2} \to \Sigma \partial(V \times I)$.

Suppose now that $\theta_1 : V \to V$ and $\theta_2 : V \to V$ are two monodromies. We shall give sufficient conditions for the associated dual homotopy Seifert pairings to be s-isometric.

3.7 **Proposition.** *Let $\Psi_1 : S^{n+2} \to V * V$ and $\Psi_2 : S^{n+2} \to V * V$ be the dual homotopy Seifert pairings associated with the monodromies θ_1 and θ_2 respectively. Then Ψ_1 is stably homotopic to Ψ_2 if θ_1 is stably homotopic to $\theta_2 \operatorname{rel} \partial V$. In particular, the identity map of V provides an s-isometry of the pairings.*

Proof. For $i=1,2$, let $\tau_i:\partial(V\times[-1,1])\to V\times V$ be the map given by the formula $\tau_i(v,t)=(v,\theta_i(v))$, if $t=1$, and $\tau_i(v,t)=(v,v)$, if $v\in\partial V$ or if $t=-1$.

We claim that τ_1 and τ_2 are stably homotopic. To see this, note that up to homotopy, $\partial(V\times[-1,1])$ is the space obtained from $V\vee V$ by attaching an $(n+1)$-cell using the map

$$S^n \xrightarrow{p_{1,-1}} S^n\vee S^n \xrightarrow{\alpha\vee\alpha} V\vee V,$$

where $\alpha:S^n\subsetneq V$ is the inclusion of the boundary of V, and $p_{1,-1}:S^n\to S^n\vee S^n$ is the "anti-pinch" map (which is characterized up to homotopy by the fact that it induces the map $x\mapsto x\oplus -x$ upon taking n-dimensional homology). The restriction of τ_i to the subspace $V\vee V$ is the map

$$(\mathrm{id}_V\vee\mathrm{id}_V,\mathrm{id}_V\vee\theta_i):V\vee V\to V\times V.$$

Since θ_1 and θ_2 are stably homotopic rel S^n, $(\mathrm{id}_V\vee\mathrm{id}_V,\mathrm{id}_V\vee\theta_1)$ is stably homotopic to $(\mathrm{id}_V\vee\mathrm{id}_V,\ \mathrm{id}_V\vee\theta_2)\,\mathrm{rel}\,S^n\vee S^n$. This shows that the stable homotopy can be extended to the top cell of $\partial(V\times[-1,1])$, proving the claim.

Now let $u_i:S^{n+2}\to\Sigma\partial(V\times[-1,1])$ be the degree one Thom-Pontryagin collapses associated to the two framed embeddings of V in S^{n+2}. By the uniqueness of embeddings in large codimensions, the normal invariant is stably unique up to homotopy. Consequently, the maps u_1 and u_2 are stably homotopic, and hence $t_1\circ u_1:S^{n+2}\to\Sigma(V\times V)$ is stably homotopic to $t_2\circ u_2:S^{n+2}\to\Sigma(V\times V)$. Finally, retracting from $\Sigma(V\times V)$ to $V*V$ yields a stable homotopy from Ψ_1 to Ψ_2, establishing the proposition. □

The importance of 3.7 is embodied in the following corollary:

3.8 **Corollary.** *Let $\Theta_1:V*V\to S^{n+2}$ and $\Theta_2:V*V\to S^{n+2}$ be the homotopy Seifert pairings associated with the monodromies θ_1 and θ_2, respectively. Then Θ_1 is isometric to Θ_2 if θ_1 is stably homotopic to $\theta_2\,\mathrm{rel}\,\partial V$.*

Proof. By 3.5 and 3.7, $\mathrm{id}_V:V\to V$ is a stable isometry from Θ_1 to Θ_2. But the homology of $V*V$ vanishes above dimension $2n+1$, so Θ_1 and Θ_2 are in the stable range. Hence $\Theta_2\circ(f\wedge f)$ is homotopic to Θ_1 if and only if it is stably homotopic. □

Finally, we give a necessary condition for the homotopy Seifert pairings of two fibred knots to be s-isometric.

3.9 **Proposition.** *Let $\theta_V:V\to V$ and $\theta_W:W\to W$ be monodromies, and let*

$$\Theta_V:V*V\to S^{n+2}\quad\textit{and}\quad\Theta_W:W*W\to S^{n+2}$$

be their associated homotopy Seifert pairings. If $f:V\to W$ is an s-isometry from Θ_V to Θ_W, then f stably conjugates θ_V to θ_W.

Proof. Let $g_\pm=q_\pm\circ f\circ p_\pm^{-1}:C_V\to C_W$, where $p_\pm:V\to C_V$ and $q_\pm:W\to C_W$ are the push maps along the positive/negative unit normals. As f is an s-isometry, $d_V\circ(\mathrm{id}_V*p_+)$ is stably homotopic to $d_W\circ(f*g_+)\circ(\mathrm{id}_V*p_+)$, where $d_V:V*C_V\to S^{n+2}$ and $d_W:W*C_W\to S^{n+2}$ are the canonical S-duality maps. By [Kl; 5.8], the same relation holds if plus is replaced by minus, i.e., $d_V\circ(\mathrm{id}_V*p_-)$ is stably homotopic to $d_W\circ(f*g_-)\circ(\mathrm{id}_V*p_-)$. Since p_+ and p_- are homotopy equivalences, it follows that d_V is stably homotopic to both $d_W \circ (f * g_+)$ and $d_W \circ (f * g_-)$. However,

$$d_W \circ (f * g_\pm) = d_W \circ (\mathrm{id}_V * g_\pm) \circ (f * \mathrm{id}_{C_V}),$$

and consequently, $d_W \circ (\mathrm{id}_V * g_+)$ is stably homotopic to $d_W \circ (\mathrm{id}_V * g_-)$. S-duality now implies that g_+ is stably homotopic to g_-. Hence,

$$f \circ \theta_V \circ f^{-1} \simeq f \circ (p_+^{-1} \circ p_-) \circ f^{-1} = (f \circ p_+^{-1}) \circ (p_- \circ f^{-1})$$

$$= q_+^{-1} \circ g_+ \circ g_-^{-1} \circ q_- \simeq_s q_+^{-1} \circ q_- \simeq \theta_W. \quad \square$$

4 Frame-spinning

4.1. *Outline of the construction.* Suppose $M^k \subsetneq S^{n+k}$ is a framed submanifold of codimension n, whose framing we denote by ϕ. We shall associate a function

$$\sigma_{M,\phi}: n\text{-knots}/\cong \to (n+k)\text{-knots}/\cong$$

defined as follows (cf. [R; Su] for details): If $K: S^n \subsetneq S^{n+2}$ is a knot, we can assume by a standard isotopy that the restriction of K to the southern hemisphere of S^n is the standard inclusion of D^n_- in S^{n+2}, and we may also assume that K maps the northern hemisphere of S^n to the northern hemisphere of S^{n+2}. Restricting K to the northern hemisphere, we in this way obtain an embedding

$$(D^n_+, S^{n-1}) \subsetneq (D^{n+2}_+, S^{n+1})$$

which is standard on S^{n-1}. Taking the product of this embedding with the identity map of M, we obtain an embedding

$$e_{M,\phi}(K): M \times D^n_+ \subsetneq M \times D^{n+2}_+$$

which is standard on $M \times S^{n-1}$.

Identifying a tubular neighborhood of M in S^{n+k} with $M \times D^n$ via the framing ϕ, we have an embedding $M \times D^n \subsetneq S^{n+k}$. By including S^{n+k} into S^{n+k+2} in the standard way, we obtain the standard inclusion of tubular neighborhoods, $M \times D^n \subsetneq M \times D^{n+2}$.

Let C be the exterior of $M \times D^n$ in S^{n+k}, and let C' be the exterior of $M \times D^{n+2}$ in S^{n+k+2}. Then the standard inclusion $S^{n+k} \subsetneq S^{n+k+2}$ decomposes as

$$\begin{array}{ccccc} S^{n+k} & = & M \times D^n & \cup & C \\ \cap & & \cap & & \cap \\ S^{n+k+2} & = & M \times D^{n+2} & \cup & C'. \end{array}$$

We modify this inclusion by replacing the standard inclusion $M \times D^n \subsetneq M \times D^{n+2}$ with the map $e_{M,\phi}(K)$. This provides us with a new embedding $S^{n+k} \subsetneq S^{n+k+2}$, which we call the *frame-spin* of K with respect to the framed embedding $(M, \phi) \subsetneq S^{n+k}$, and which we denote by $\sigma_{M,\phi}(K)$. The procedure clearly depends only on the isotopy class of K, since the choices used to define it vary in an isotopic way. In the case $M = S^k$ with trivial framing, this is just the superspinning construction of Cappell [C].

It can be shown [Su] that the exterior of $\sigma_{M,\phi}(K)$ in S^{n+k+2} is precisely the manifold

$$X(\sigma_{M,\phi}(K)) = (D^{n+k+1} - \mathrm{int}(M \times D^n \times I)) \times S^1 \cup_{M \times D^n \times S^1} M \times X,$$

where $X = X(K)$, $S^1 \subsetneq X$ is the meridian, and $M \times D^n \times 0 \times S^1$ is identified with $M \times D^n_- \times S^1$. As the space $D^{n+k+1} - \mathrm{int}(M \times D^n \times I)$ is contractible, one easily sees that $X(\sigma_{M,\phi}(K))$ is homotopy equivalent to the mapping torus

$$(M_+ \wedge \tilde{X}) \times_{\mathrm{id}_{M_+} \wedge t} S^1,$$

where $\tilde{X}$ is the universal abelian cover of X and $t:\tilde{X} \to \tilde{X}$ is the generator of the group of covering translations corresponding to the meridian.

4.2. *Frame-spun Seifert surfaces.* If $V^{n+1} \subsetneqq S^{n+2}$ is a Seifert surface for a knot $K: S^n \subsetneqq S^{n+2}$, then a relative version of the above construction provides us with a Seifert surface for $\sigma_{M,\phi}(K)$, which is unique up to isotopy. This procedure carries isotopy classes of Seifert surfaces for K into isotopy classes of Seifert surfaces for $\sigma_{M,\phi}(K)$.

Define

$$\sigma_{M,\phi}(V) = (D^{n+k+1} - \mathrm{int}(M \times D^n \times I)) \cup_{M \times D^n} M \times V.$$

Then $\sigma_{M,\phi}(V)$ is the Seifert surface for $\sigma_{M,\phi}(K)$. The homotopy type of this Seifert surface is given by the following lemma.

4.3 **Lemma.** *There is a homotopy equivalence $\sigma_{M,\phi}(V) \simeq M_+ \wedge V$. With respect to this identification, the inclusion $\partial\sigma_{M,\phi}(V) \subsetneqq \sigma_{M,\phi}(V)$ corresponds to the composite*

$$(\mathrm{id}_{M_+} \wedge \alpha) \circ u : S^{n+k} \to M_+ \wedge V,$$

where $\alpha: S^n \subsetneqq V$ is the inclusion, and where $u: S^{n+k} \to M_+ \wedge S^n$ is the Thom-Pontryagin collapse of the framed embedding $(M, \phi) \subsetneqq S^{n+k}$.

Proof. Since $D^{n+k+1} - \mathrm{int}(M \times D^n \times I)$ is contractible, $\sigma_{M,\phi}(V)$ is homotopy equivalent to $M \times V/M \times * = M_+ \wedge V$. The inclusion $\partial\sigma_{M,\phi}(V) \subsetneqq \sigma_{M,\phi}(V)$ is by definition

$$\begin{aligned} S^{n+k} &= (S^{n+k} - \mathrm{int}(M \times D^n \times 0)) \cup_{M \times S^{n-1}} M \times D^n \\ &\subsetneqq (D^{n+k+1} - \mathrm{int}(M \times D^n \times I)) \cup_{M \times D^n} M \times V, \end{aligned}$$

which, under the identification $\sigma_{M,\phi}(V) \simeq M_+ \wedge V$ is the map

$$\begin{aligned} S^{n+k} \to S^{n+k}/(S^{n+k} - \mathrm{int}(M \times D^n \times 0)) &= M \times D^n/M \times S^{n-1} \\ &\subsetneqq M \times V/M \times D^n \simeq M_+ \wedge V. \end{aligned}$$

As $M \times D^n/M \times S^{n-1} \subsetneqq M \times V/M \times D^n$ is equated with the inclusion

$$M_+ \wedge S^n \subsetneqq M_+ \wedge V,$$

we are done. □

Finally, we remark that if $K: S^n \subsetneqq S^{n+2}$ is a fibred knot with fibreing $p: X(K) \to S^1$, with fibre V and monodromy $\theta: V \to V$, then $\sigma_{M,\phi}(K)$ is a fibred knot with fibration

$$\mathrm{pr}_2 \cup p \circ \mathrm{pr}_2 : (D^{n+k+1} - \mathrm{int}(M \times D^n \times I)) \times S^1 \cup_{M \times D^n \times S^1} M \times X \to S^1,$$

with fibre $\sigma_{M,\phi}(V)$ and monodromy $\sigma_{M,\phi}(\theta): \sigma_{M,\phi}(V) \to \sigma_{M,\phi}(V)$ given by

$$\begin{aligned} \mathrm{id} \cup (\mathrm{id} \times \theta) : (D^{n+k+1} &- \mathrm{int}(M \times D^n \times I)) \cup_{M \times D^n} M \times V \\ &\to (D^{n+k+1} - \mathrm{int}(M \times D^n \times I)) \cup_{M \times D^n} M \times V. \end{aligned}$$

Up to homotopy (Lemma 4.3), this is the map $\mathrm{id}_{M_+} \wedge \theta: M_+ \wedge V \to M_+ \wedge V$.

4.4. *Knots with isometric homology Seifert pairings but non-isometric homotopy Seifert pairings.* We use frame-spinning to give an example of a pair of 1-simple fibred knots whose exteriors are not homotopy equivalent, whose homotopy Seifert pairings are not isometric (in fact, not even s-isometric), yet whose homology Seifert pairings are isometric.

Let $SU(3)$ be the 8-dimensional Lie group of 3×3 Hermitian matrices with determinant one. Construct a framed embedding $e: SU(3) \subsetneq S^{11}$ as follows: Let $e_0: SU(3) \subsetneq S^5 \times S^5$ be the embedding given by sending a matrix to its last two column vectors. Then e is defined to be e_0 followed by the standard inclusion $S^5 \times S^5 \subsetneq S^{11}$. Since the normal bundle of e_0 is codimension two and oriented, it is trivial, hence the normal bundle of e is also trivial.

Let $K: S^3 \subsetneq S^5$ be any non-trivial simple fibred 3-knot with fibre V and monodromy $\theta: V \to V$. Let $K_1 = \sigma_{S^3\times S^5}(K)$, the frame-spin of K with respect to the trivial framed embedding $S^3 \times S^5 \subsetneq S^{11}$, and let $K_2 = \sigma_{SU(3)}(K)$ the frame-spin of K with respect to the framed embedding $e: SU(3) \subsetneq S^{11}$.

4.5 **Theorem.** *The* 1*-simple fibred* 11*-knots* K_1 *and* K_2 *satisfy the following:*

(1) *the homology Seifert pairings of* K_1 *and* K_2 *are isometric;*

(2) *the homotopy Seifert pairings of* K_1 *and* K_2 *are not s-isometric;*

(3) *the exteriors of* K_1 *and* K_2 *have non-isomorphic stable homotopy groups.*

Proof. (1) From the Gysin sequence of the principal fibration $S^3 \to SU(3) \to S^5$, it follows that $SU(3)$ and $S^3 \times S^5$ have isomorphic homology. Let $\Phi: H_*(S^3 \times S^5) \to H_*(SU(3))$ be this isomorphism. The Künneth formula then implies that

$$\Phi_+ \otimes \mathrm{id}_{H_*(V)}: H_*((S^3 \times S^5)_+ \wedge V) \to H_*(SU(3)_+ \wedge V)$$

is an isomorphism. As the homology monodromy of K_1 is $\mathrm{id}_{H_*(SU(3)_+)} \otimes \theta_*$, and that of K_2 is $\mathrm{id}_{H_*(S^3\times S^5)_+} \otimes \theta_*$, it follows that $\Phi_+ \otimes \mathrm{id}_{H_*(V)}$ conjugates the former to the latter. Hence the homology Seifert pairings of K_1 and K_2 are isometric.

(2) and (3): To show that the homotopy Seifert pairings of K_1 and K_2 are not s-isometric, it is sufficient to show by 3.9 that the monodromies of K_1 and K_2 are not stably homotopy conjugate. But this will be true if the fibres of K_1 and K_2 have non-isomorphic stable homotopy groups, and this will also prove (3).

As V is a wedge of 2-spheres, the stable homotopy group $\pi_6^s((S^3 \times S^5)_+ \wedge V)$ is a direct sum of copies of $\pi_4^s(S^3 \times S^5)_+$, and an elementary computation shows that this last group is isomorphic to $\mathbb{Z}_2$. Similarly, $\pi_6^s(SU(3)_+ \wedge V)$ is a direct sum of copies of $\pi_4^s(SU(3)_+)$. But $\pi_4^s(SU(3)_+)$ is trivial by the Gysin sequence in stable homotopy. Hence the fibres of K_1 and K_2 have different stable homotopy groups. □

As a final remark, we note that varying the choice of the simple 3-knot K (as in 2.5), we obtain infinitely many pairs of knots in dimension 11 satisfying the conclusions of 4.5. By superspinning these knots, we obtain infinitely many pairs of 1-simple fibred n-knots, $n \geq 11$, satisfying the conclusions of 4.5.

5 Diff-spinning

5.1. *The construction.* In this section we modify the frame-spinning construction by twisting the framed submanifold by a self diffeomorphism as we spin. This modification does not change the universal abelian cover of the frame-spun knot, but composes the action of the meridian on the universal abelian cover with a certain diffeomorphism.

Let (M^k, ϕ) be a framed submanifold of S^{n+k}, and let $M \times D^n$ be identified with a tubular neighborhood of M via ϕ. Assume that we are given a diffeomorphism $\eta: M \to M$ satisfying the following properties:

(d_1) there is a diffeomorphism $\tilde{\eta}: S^{n+k} \to S^{n+k}$ such that $\tilde{\eta}_{|M \times D^n} = \eta \times \mathrm{id}_{D^n}$;
(d_2) $\tilde{\eta}$ is isotopic to the identity;
(d_3) $\eta_*: H_*(M) \to H_*(M)$ is the identity.

Note that condition (d_3) is certainly implied by the condition

(d'_3) the map $\eta: M \to M$ is stably homotopic to the identity.

Given such η, we fix the extension $\tilde{\eta}$ and its isotopy to the identity as part of the data. Considering S^{n+k} as the boundary of the disk D^{n+k+1}, we obtain an embedding of $M \times D^n \times I$ in D^{n+k+1}. By (d_1) and (d_2), the diffeomorphism $\tilde{\eta}$ has a preferred extension to a diffeomorphism $\bar{\eta}: D^{n+k+1} \to D^{n+k+1}$ that preserves $M \times D^n \times I$.

Given a knot $K: S^n \subsetneq S^{n+2}$, a framed submanifold $(M^k, \phi) \subsetneq S^{n+k}$, and a diffeomorphism $\eta: M \to M$ as above, we define the *η-modification* of $\sigma_{M,\phi}(K)$ by first forming the "twisted" exterior,

$$\eta \bullet X(\sigma_{M,\phi}(K)) = (D^{n+k+1} - \mathrm{int}(M \times D^n \times I)) \times_{\bar{\eta}} S^1 \cup_{(M \times D^n) \times_{\eta \times \mathrm{id}} S^1} M \times_\eta X,$$

where $M \times_\eta X$ is the space $M \times_{\mathbb{Z}} \tilde{X}$, with $\mathbb{Z}$ acting on M via the diffeomorphism η and on $\tilde{X}$ via the covering translation t. Note that $\eta \bullet X(\sigma_{M,\phi}(K))$ has the homotopy type of the mapping torus

$$(M_+ \wedge \tilde{X}) \times_{\eta_+ \wedge t} S^1.$$

5.2 **Lemma.** (1) *The manifold $\eta \bullet X(\sigma_{M,\phi}(K))$ is a homology circle.*

(2) *The fundamental group of $\eta \bullet X(\sigma_{M,\phi}(K))$ is the same as the fundamental group of $X(K)$.*

Proof. (1) Follows from the Wang sequence of the fibration $(M_+ \wedge \tilde{X}) \times_{\eta_+ \wedge t} S^1 \to S^1$ and condition (d_3).

(2) As $D^{n+k+1} - \mathrm{int}(M \times D^n \times I)$ is contractible, $\eta \bullet X(\sigma_{M,\phi}(K))$ is homotopy equivalent to the space $* \times S^1 \cup_{M \times_\eta S^1} M \times_\eta X$. The projection onto the second coordinate gives a map

$$p_2: * \times S^1 \cup_{M \times_\eta S^1} M \times_\eta X \to S^1 \cup_{S^1} X = X.$$

The Van-Kampen theorem then shows that p_2 induces an isomorphism of fundamental groups. □

By definition, the boundary of the twisted exterior is $\eta \bullet \partial X(\sigma_{M,\phi}(K)) = S^{n+k} \times_{\tilde{\eta}} S^1$. Using the give isotopy of $\tilde{\eta}$ to the identity, $S^{n+k} \times_{\tilde{\eta}} S^1$ becomes identified with $S^{n+k} \times S^1$. Gluing on a copy $S^{n+k} \times D^2$ to $\eta \bullet X(\sigma_{M,\phi}(K))$, we obtain a knot in a homotopy sphere Σ^{n+k+2} by 5.2. Changing the smooth structure of Σ if necessary to get a standard $(n+k+2)$-sphere, we obtain a knot $\eta \bullet \sigma_{M,\phi}(K): S^{n+k} \subsetneq S^{n+k+2}$.

5.3 **Definition.** The η-modified frame-spun knot, $\eta \bullet \sigma_{M,\phi}(K): S^{n+k} \subsetneq S^{n+k+2}$, will be called the *diff-spin* of $K: S^n \subsetneq S^{n+2}$ with respect to the pair $((M, \phi), \eta)$.

5.4. *The diff-spun monodromy.* Let $K: S^n \subsetneq S^{n+2}$ be a fibred knot with fibre V^{n+1} and monodromy θ. Then the diff-spun knot $\eta \bullet \sigma_{M,\phi}(K)$ is also fibred. It has the same fibre as $\sigma_{M,\phi}(K)$, i.e.,

$$\eta \bullet \sigma_{M,\phi}(V) = (D^{n+k+1} - \mathrm{int}(M \times D^n \times I)) \cup_{M \times D^n} M \times V.$$

However, $\eta \bullet \sigma_{M,\phi}(K)$ has monodromy

$$\eta \bullet \sigma_{M,\phi}(\theta) = \bar{\eta} \cup \eta \times \theta.$$

Using the identification $\sigma_{M,\phi}(V) \simeq M_+ \wedge V$ (Lemma 4.3), the diff-spun monodromy $\eta \bullet \sigma_{M,\phi}(\theta)$ is identified with the map $\eta_+ \wedge \theta : M_+ \wedge V \to M_+ \wedge V$.

5.5. *Frame-spinning the diff-spin construction.* Let (M^k, ϕ) be a framed submanifold of $D^{n+k} \subsetneq S^{n+k}$, and let $M \times D^n$ be identified with a tubular neighborhood of M via ϕ. Assume that we are given a diffeomorphism $\eta : M \to M$ satisfying the conditions (d_1)–(d_3). Suppose further that $(N^j, \psi) \subsetneq S^{n+k+j}$ is a framed submanifold, and let $N \times D^{n+k}$ be identified with a tubular neighborhood of N via ψ. The composition $N \times M \subsetneq N \times M \times D^n \subsetneq N \times D^{n+k} \subsetneq S^{n+k+j}$ is an embedding with framing $\psi \otimes \phi = \psi \circ (\mathrm{id}_N \times \phi)$.

Consider the diffeomorphism $\eta_N = \mathrm{id}_N \times \eta : N \times M \to N \times M$.

5.6 **Lemma.** *With $(N^j \times M^k, \psi \otimes \phi) \subsetneq S^{n+k+j}$ as above, the diffeomorphism $\eta_N : N \times M \to N \times M$ satisfies conditions* (d_1)–(d_3). *If $\eta : M \to M$ satisfies* (d_3'), *then so does η_N.*

Proof. Conditions (d_1) and (d_2): Let $\tilde{\eta}_t : S^{n+k} \to S^{n+k}$ be the isotopy of the identity to $\tilde{\eta}$, the extension of η to S^{n+k}. Then $\tilde{\eta}_t$ determines a diffeomorphism $F : D^{n+k+1} \to D^{n+k+1}$ whose formula in polar coordinates is $F(t \cdot x) = t \cdot \tilde{\eta}_t(x)$, for $x \in S^{n+k}$ and $t \in I$ (note that the derivative of F at the origin is the identity). Consider the isotopy $\tilde{\eta}_{N_t}$ of $S^{n+k+j} = \partial(D^{n+k+1} \times D^j)$ defined by the formula

$$\tilde{\eta}_{N_t}(x_1, \ldots, x_{n+k+j+1}) = (x_1, \ldots, x_j, F(t \cdot (x_{j+1}, \ldots, x_{n+k+j+1}))/t).$$

Then $\tilde{\eta}_{N_0} = \mathrm{id}_{S^{n+k+j}}$ and $\tilde{\eta}_{N_1} : S^{n+k+j} \to S^{n+k+j}$ is an extension of $\eta_N : N \times M \to N \times M$. Conditions (d_3) and (d_3'): These are trivial. □

The lemma above shows that the η_N-modification of $\sigma_{N \times M, \psi \otimes \phi}(K)$ is defined. In fact it is easy to see that the knot $\eta_N \bullet \sigma_{N \times M, \psi \otimes \phi}(K)$ is equivalent to the knot $\sigma_{N,\psi}(\eta \bullet \sigma_{M,\phi}(K))$.

6 Comparison of frame-spinning to diff-spinning

6.1. We now compare the two constructions, frame-spinning and diff-spinning. We shall consider the case of fibred knots only.

Let $K : S^n \subsetneq S^{n+2}$ be a fibred knot with fibre V^{n+1} and monodromy θ. Let (M^k, ϕ) be a framed submanifold of S^{n+k}, and let η be a diffeomorphism of M satisfying conditions (d_1), (d_2), and (d_3') of Sect. 5. Consider the $(n+k)$-knots $K_1 = \sigma_{M,\phi}(K)$ and $K_2 = \eta \bullet \sigma_{M,\phi}(K)$. By the remarks of Sects. 4 and 5, both knots are fibred, with fibre $\sigma_{M,\phi}(V)$ and monodromies $\theta_1 = \sigma_{M,\phi}(\theta)$ and $\theta_2 = \eta \bullet \sigma_{M,\phi}(\theta)$, respectively.

The following proposition shows that the monodromy of K_1 is stably homotopic to the monodromy of K_2 relative to $S^{n+k} = \partial \sigma_{M,\phi}(V)$.

6.2 **Proposition.** *The monodromies θ_1 and θ_2 are stably homotopic rel boundary.*

Proof. Using the identification $\sigma_{M,\phi}(V) \simeq M_+ \wedge V$ of 4.3, we must show that

$$\mathrm{id}_{M_+} \wedge \theta : M_+ \wedge V \to M_+ \wedge V$$

is stably homotopic to $\eta_+ \wedge \theta: M_+ \wedge V \to M_+ \wedge V$ relative to the map $\omega: S^{n+k} \to M_+ \wedge V$, given by the Thom-Pontryagin construction $u: S^{n+k} \to M_+ \wedge S^n$ followed by the inclusion $M_+ \wedge S^n \subsetneqq M_+ \wedge V$.

Consider the space $M_+ \wedge V \cup_\omega D^{n+k+1}$ obtained by attaching an $(n+k+1)$-cell to $M_+ \wedge V$ along ω. Consider also the space $M_+ \wedge (V \cup_\alpha D^{n+1})$, where $\alpha: S^n \subsetneqq V$ is the inclusion of the boundary. There is then a natural "collapse" map from the cofibre of ω to the cofibre of α, denoted by

$$c: M_+ \wedge V \cup_\omega D^{n+k+1} \to M_+ \wedge (V \cup_\alpha D^{n+1}).$$

The map c has a stable homotopy retraction arising from the facts

(i) $$M_+ \wedge V \cup_\omega D^{n+k+1} \simeq_s (M_+ \wedge V) \vee S^{n+k+1},$$

(ii) $$M_+ \wedge (V \cup_\alpha D^{n+1}) \simeq_s (M_+ \wedge V) \vee (M_+ \wedge S^{n+1}),.$$

and (iii) the Thom-Pontryagin map $u: S^{n+k} \to M_+ \wedge S^{n+1}$ is a homotopy retract. The collapse map is natural in the following sense: For any map $f: M \to M$, there is a commutative diagram

$$\begin{tikzcd}
M_+ \wedge V \cup_\omega D^{n+k+1} \arrow[r, "(f_+ \wedge \theta) \cup \mathrm{id}"] \arrow[d, "c"] & M_+ \wedge V \cup_\omega D^{n+k+1} \arrow[d, "c"] \\
M_+ \wedge (V \cup_\alpha D^{n+1}) \arrow[r, "f_+ \wedge (\theta \cup \mathrm{id})"'] & M_+ \wedge (V \cup_\alpha D^{n+1}).
\end{tikzcd}$$

Since $\eta: M \to M$ is stably the identity by (d_3'), the map $\eta_+ \wedge (\theta \cup \mathrm{id})$ is stably homotopic to $\mathrm{id}_{M_+} \wedge (\theta \cup \mathrm{id})$. By naturality, $c \circ ((\eta_+ \wedge \theta) \cup \mathrm{id}) = (\eta_+ \wedge (\theta \cup \mathrm{id})) \circ c$ is stably homotopic to

$$(\mathrm{id}_{M_+} \wedge (\theta \cup \mathrm{id})) \circ c = c \circ ((\mathrm{id}_{M_+} \wedge \theta) \cup \mathrm{id}).$$

As c has a stable retraction, it follows that $(\mathrm{id}_{M_+} \wedge \theta) \cup \mathrm{id}$ is stably homotopic to $(\eta_+ \wedge \theta) \cup \mathrm{id}$. Since stable homotopy is excisive, it now follows that $\mathrm{id}_{M_+} \wedge \theta: M_+ \wedge V \to M_+ \wedge V$ is stably homotopic to $\eta_+ \wedge \theta: M_+ \wedge V \to M_+ \wedge V$ relative to the map $\omega: S^{n+k} \to M_+ \wedge V$. □

Combining 3.8 with 6.2, we deduce the following theorem.

6.3 **Theorem.** *The frame-spun knot K_1 and the diff-spun knot K_2 defined above have isometric homotopy Seifert pairings.*

6.4. *Conditions which guarantee that the frame-spun and diff-spun complements are homotopically inequivalent.* Assume that $K: S^n \subsetneqq S^{n+2}$ is a non-trivial simple fibred n-knot with $n = 2m - 1$. Then the fibre V is homotopy equivalent to a wedge of m-spheres. Assume further that the monodromy $\theta: V \to V$ satisfies [in addition to (m_1)–(m_3) of Sect. 2], the condition

(m_4) θ has homotopy order $c > 1$.

Suppose also that the diffeomorphism $\eta: M \to M$ satisfies [in addition to (d_1)–(d_3) of Sect. 5], the condition

(d_4) $\Sigma^m \eta: \Sigma^m M \to \Sigma^m M$ has homotopy order $d > 1$.

We then have the following theorem.

6.5 **Theorem.** *If the integers c and d are coprime, then the exteriors of $K_1 = \sigma_{M,\phi}(K)$ and $K_2 = \eta \bullet \sigma_{M,\phi}(K)$ are not homotopy equivalent.*

Proof. By 2.3, it is sufficient to show that the monodromies θ_1 of K_1 and θ_2 of K_2 are not homotopically conjugate. By Sects. 4 and 5, the fibres of K_1 and K_2 are homotopy equivalent to $M_+ \wedge V$, and with respect to these identifications, $\theta_1 = \mathrm{id}_{M_+} \wedge \theta$ and $\theta_2 = \eta_+ \wedge \theta$.

Suppose there existed a homotopy equivalence $f: M_+ \wedge V \to M_+ \wedge V$ such that $f \circ \theta_1 \circ f^{-1} \simeq \theta_2$. Then $f \circ \theta_1^c \circ f^{-1} \simeq \theta_2^c$. As $\theta^c \simeq \mathrm{id}_V$, by assumption, we get

$$\eta_+^c \wedge \mathrm{id}_V \simeq \mathrm{id}_{M_+ \wedge V}.$$

Choose a homotopy F from $\eta_+^c \wedge \mathrm{id}_V$ to $\mathrm{id}_{M_+ \wedge V}$. Let $\iota: S^m \to V$ be the inclusion of one of the factors of the wedge decomposition of V, and let $\pi: V \to S^m$ be the projection on that factor. Then the composite

$$(M_+ \wedge S^m) \times I \xrightarrow{(\mathrm{id}_{M_+} \wedge \iota) \times \mathrm{id}_I} (M_+ \wedge V) \times I \xrightarrow{F} M_+ \wedge V \xrightarrow{\mathrm{id}_{M_+} \wedge \pi} M_+ \wedge S^m$$

is a homotopy between $\eta_+^c \wedge \mathrm{id}_{S^m}$ and $\mathrm{id}_{M_+ \wedge S^m}$. Consequently,

$$\Sigma^m \eta^c \simeq \mathrm{id}_{\Sigma^m M}.$$

On the other hand, we have by assumption $\Sigma^m \eta^d \simeq \mathrm{id}_{\Sigma^m M}$. As $(c, d) = 1$, it follows that $\Sigma^m \eta \simeq \mathrm{id}_{\Sigma^m M}$, which contradicts ($d_4$). $\square$

7 Diffeomorphisms of $S^p \times S^q$

7.1. In this section we construct self-diffeomorphisms of $S^p \times S^q$ that have a suspension of finite non-trivial homotopy order, yet which are stably homotopic to the identity. Let $SO(k)$ be the Lie group of $k \times k$ orthogonal matrices with determinant one. The basepoint is the identity matrix I_k. Let $\cdot$ denote the linear action of $SO(k)$ on S^{k-1} and $i_{k,l}: SO(k) \subsetneq SO(l)$ the standard inclusion.

Suppose $\lambda: S^p \to SO(q+1)$ is a pointed map, where p and q are positive. The *adjoint* of λ is the map $\bar{\lambda}: S^p \times S^q \to S^q$ given by $\bar{\lambda}(x, y) = \lambda(x) \cdot y$. We let the map $J(\lambda): S^{p+q+1} \to S^{q+1}$ be the Hopf construction of $\bar{\lambda}$, $H(\bar{\lambda}): S^p * S^q \to \Sigma S^q$. If $\beta: S^k \to S^p$ is a map, then the following hold: $J(\lambda \circ \beta) = J(\lambda) \circ \Sigma^{q+1} \beta$ and $J(i_{q+1,q+n} \circ \lambda) = \Sigma^{n-1} J(\lambda)$. The assignment $\lambda \mapsto J(\lambda)$ is well-known to induce the J-homomorphism $\pi_p(SO(q+1)) \to \pi_{p+q+1}(S^{q+1})$ (cf. [Wh]).

Associated to $\lambda: S^p \to SO(q+1)$, there is a diffeomorphism $g(\lambda): S^p \times S^q \to S^p \times S^q$ given by

$$g(\lambda)(x, y) = (x, \bar{\lambda}(x, y)).$$

The maps $J(\lambda)$ and $g(\lambda)$ are related by the formula $J(\lambda) = H(\mathrm{pr}_2 \circ g(\lambda))$, where $\mathrm{pr}_2: S^p \times S^q \to S^q$ is the second coordinate projection.

7.2 **Proposition.** *For a positive integer n, $\Sigma^n g(\lambda): \Sigma^n(S^p \times S^q) \to \Sigma^n(S^p \times S^q)$ is homotopic to the identity if and only if $\Sigma^{n-1} J(\lambda): S^{p+q+n} \to S^{q+n}$ is null-homotopic.*

Proof. We calculate the obstruction for $\Sigma^n g(\lambda)$ to be homotopic to the identity and show that it is precisely $\Sigma^{n-1} J(\lambda)$. Consider the restriction of $g(\lambda)$ to $S^p \vee S^q$. This restriction is equal to the inclusion $S^p \vee S^q \subsetneq S^p \times S^q$, since

$$g(\lambda)(x, *) = (x, \lambda(x) \cdot *) = (x, *), \quad \text{and} \quad g(\lambda)(*, y) = (*, \lambda(*) \cdot y) = (*, I_{q+1} \cdot y) = (*, y).$$

Consequently, $\Sigma^n g(\lambda)$ restricted to $\Sigma^n(S^p \vee S^q)$ is equal to the inclusion

$$\Sigma^n(S^p \vee S^q) \subsetneq \Sigma^n(S^p \times S^q).$$

To complete the proof it will be sufficient to show that the only obstruction to extending the homotopy to the top cell of $\Sigma^n(S^p \times S^q)$ is the homotopy class of $\Sigma^{n-1}J(\lambda)$.

This obstruction lives in the group $\pi_{p+q+n}(\Sigma^n(S^p \times S^q))$. By the Blakers-Massey excision theorem (see [Wh, p. 366]), this group is naturally isomorphic to the direct sum

$$\pi_{p+q+n}(\Sigma^n S^p) \oplus \pi_{p+q+n}(\Sigma^n S^q) \oplus \pi_{p+q+n}(\Sigma^n S^p \wedge S^q).$$

As may be readily calculated, the projection of the obstruction onto the first group is just the homotopy class of the map

$$S^{p+q+n} \simeq \Sigma^{n-1}(S^p * S^q) \xrightarrow{\Sigma^{n-1}H(\mathrm{id})} \Sigma^n(S^p \times S^q) \xrightarrow{\Sigma^n \mathrm{pr}_1} \Sigma^n S^p,$$

where id is the identity of $S^p \times S^q$. By 7.1, this map is

$$\Sigma^{n-1}(\Sigma\, \mathrm{pr}_1 \circ H(\mathrm{id})) = \Sigma^{n-1}H(\mathrm{pr}_1),$$

which is null homotopic, since pr_1 factors through $S^p \times *$.

The projection of the obstruction onto the second group is similarly the homotopy class of the map

$$S^{p+q+n} \simeq \Sigma^{n-1}(S^p * S^q) \xrightarrow{\Sigma^{n-1}H(\mathrm{id})} \Sigma^n(S^p \times S^q) \xrightarrow{\Sigma^n(\mathrm{pr}_2 \circ g(\lambda))} \Sigma^n S^q.$$

By 1.1 and 7.1, this map is

$$\Sigma^{n-1}(\Sigma(\mathrm{pr}_2 \circ g(\lambda)) \circ H(\mathrm{id})) = \Sigma^{n-1}(H(\mathrm{pr}_2 \circ g(\lambda)) = \Sigma^{n-1}J(\lambda).$$

Lastly, the projection of the obstruction onto the third group is the difference between the identity map of S^{p+q+n} and the composite

$$S^{p+q+n} \simeq \Sigma^{n-1}(S^p * S^q) \xrightarrow{\Sigma^{n-1}H(\mathrm{id})} \Sigma^n(S^p \times S^q) \xrightarrow{\Sigma^n \pi} \Sigma^n(S^p \wedge S^q).$$

By 1.1, this map has degree one, and so the difference in question is null-homotopic. This completes the proof of the proposition. $\square$

We consider pointed maps $\lambda: S^p \to SO(q+1)$ which satisfy the following condition:

(r_1) for some integer $n > 1$, the composite $i_{q+1,q+n} \circ \lambda: S^p \to SO(q+n)$ is null-homotopic.

Decompose S^{p+q+n} into the union of two solid tori $S^p \times D^{q+n} \cup D^{p+1} \times S^{q+n-1}$, with $S^p \times S^q$ embedded standardly inside $S^p \times D^{q+n}$. We then have:

7.3 Proposition. *If $\lambda: S^p \to SO(q+1)$ satisfies condition* (r_1), *then the associated diffeomorphism $g(\lambda): S^p \times S^q \to S^p \times S^q$ satisfies conditions* (d_1), (d_2), *and* (d'_3) *of Sect. 5.*

Proof. We first extend $g(\lambda)$ to S^{p+q+n}, and then prove that the extension is isotopic to the identity. This will establish (d_1) and (d_2). Condition (d'_3) follows immediately from the equation $J(i_{q+1,q+n} \circ \lambda) = \Sigma^{n-1}J(\lambda)$ and 7.2.

For the first part, choose a pointed null-homotopy

$$F: D^{p+1} \to SO(q+n) \quad \text{of} \quad i_{q+1,q+n} \circ \lambda,$$

as guaranteed by (r_1). Define an extension $\gamma: S^{p+q+n} \to S^{p+q+n}$ of $g(\lambda)$ by the formula

$$\gamma(x,v) = (x, F(x) \cdot v), \quad \text{if} \quad (x,v) \in S^p \times D^{q+n} \quad \text{or if} \quad (x,v) \in D^{p+1} \times S^{q+n-1}.$$

The isotopy $\gamma_t: S^{p+q+n} \to S^{p+q+n}$ from γ to the identity is given by the formula $\gamma_t(x, v) = (x, F((1-t)x) \cdot v)$. □

Finally, we shall consider pointed maps $\lambda: S^p \to SO(q+1)$ which satisfy

(r_2) for some integer $m > 1$, the map $\Sigma^{m-1}J(\lambda): \Sigma^{m-1}S^p * S^q \to \Sigma^{m-1}S^{q+1}$ is essential.

7.4 **Proposition.** *If the map $\lambda: S^p \to SO(q+1)$ satisfies condition* (r_2), *then the diffeomorphism $g(\lambda): S^p \times S^q \to S^p \times S^q$ satisfies condition* (d_4) *of Sect. 6.*

Proof. By 7.2, $\Sigma^m g(\lambda)$ is not homotopic to the identity. The homotopy class of $\Sigma^{m-1}J(\lambda)$ is an element of $\pi_{p+q+m}(S^{q+m})$, which, by a theorem of Serre, is a finite group, unless $m = p - q + 1$ with p odd. In that case, the homotopy class of $\Sigma^{m-2}J(\lambda)$ is an element of the finite group $\pi_{2p}(S^p)$, and consequently the homotopy class of $\Sigma^{m-1}J(\lambda) = \Sigma\Sigma^{m-2}J(\lambda)$ is of finite order. Hence, the homotopy class of $\Sigma^{m-1}J(\lambda)$ always has finite order. Thus by 7.2 again, $\Sigma^m g(\lambda)$ has finite homotopy order. □

We now exhibit maps $\lambda: S^p \to SO(q+1)$ satisfying conditions (r_1) and (r_2). Notice that for these conditions to be satisfied simultaneously, we must have $q > 1$, $m < n$, and $m \leqq p - q$.

Recall that the fibre bundle $SO(q) \to SO(q+1) \to S^q$ has a section $\mu_q: S^q \to SO(q+1)$ if $q = 1, 3$, or 7. Moreover, $J(\mu_1) = \eta_2: S^3 \to S^2$, $J(\mu_3) = \nu_4: S^7 \to S^4$, and $J(\mu_7) = \sigma_8: S^{15} \to S^8$ are the Hopf fibrations.

7.5 *Example.* With $p = 4$, $q = 2$, $n = 3$, and $m = 2$. Let $\tau: S^3 \to SO(3)$ be the universal cover. We define $\lambda: S^4 \to SO(3)$ to be the composite

$$S^4 \xrightarrow{\eta_3} S^3 \xrightarrow{\tau} SO(3).$$

Condition (r_1). The composite $i_{3,5} \circ \lambda: S^4 \to SO(5)$ is null-homotopic, since the homomorphism $i_{3,5*}: \pi_4(SO(3)) \to \pi_4(SO(5))$ is zero (see [Wh, p. 200]).

Condition (r_2). We have $J(\lambda) = J(\tau \circ \eta_3) = J(\tau) \circ \Sigma^3\eta_3 = \nu' \circ \eta_6$, where ν' is the clutching map for the principal bundle $S^3 \to Sp(2) \to S^7$. Then $\Sigma J(\lambda) = \Sigma\nu' \circ \eta_7$, and this has homotopy order 2 [T, p. 42–43].

7.6 *Example.* With $p = 10$, $q = 7$, $n = 5$, and $m = 3$. We define $\lambda: S^{10} \to SO(8)$ to be the composite

$$S^{10} \xrightarrow{\nu_7} S^7 \xrightarrow{\mu_7} SO(8).$$

Condition (r_1). We must show that $i_{8,12} \circ \lambda: S^{10} \to SO(12)$ is null-homotopic. By stability, $\pi_{10}(SO(12)) \cong \pi_{10}(SO)$. By Bott periodicity, $\pi_{10}(SO) \cong \pi_2(SO) = 0$.

Condition (r_2). We have $J(\lambda) = J(\mu_7 \circ \nu_7) = J(\mu_7) \circ \Sigma^8\nu_7 = \sigma_8 \circ \nu_{15}$. Then $\Sigma^2 J(\lambda) = \sigma_{10} \circ \nu_{17}$, and this has even homotopy order by Toda [T, p. 66].

7.7 *Example.* With $p = 10$, $q = 4$, $n = 11$, and $m = 6$. Let $\upsilon: Sp(2) \to SO(5)$ be the universal cover [Wh, p. 715]. Let $\varrho: S^{10} \to Sp(2)$ be the clutching map for the principal bundle $Sp(2) \to Sp(3) \to S^{11}$. We define $\lambda: S^{10} \to SO(5)$ to be the composite

$$S^{10} \xrightarrow{\varrho} Sp(2) \xrightarrow{\upsilon} SO(5).$$

Condition (r_1). We must show that $i_{5,11} \circ \lambda: S^{10} \to SO(11)$ is null-homotopic. This follows from the fact that $\pi_{10}(SO(11))$ has order 2 (see [Ke]), and the formula $i_{5,9} \circ \upsilon \circ \varrho \simeq 2 i_{8,9} \circ \mu_7 \circ \nu_7$ from [O].

Condition (r_2). From [O], we have $J(\lambda)=\nu_5\circ\sigma_8$. Then $\Sigma^5 J(\lambda)=\nu_{10}\circ\sigma_{13}$, and this map has even homotopy order [T, p. 66–72].

8 The examples

8.1. Let $K=K_{a,b}:S^n\subsetneq S^{n+2}$ be the simple fibred n-knot, $n=2m-1$, whose monodromy has homotopy order $c=ab$ given by Proposition 2.5. Set M equal to $S^p\times S^q$, standardly embedded in S^{p+q+n}. Let $L=\sigma_M(K)$ be the frame-spin of K with respect to the trivial framing. Let $\lambda:S^p\to SO(q+1)$ satisfy (r_1) and (r_2). Set $\eta=g(\lambda):S^p\times S^q\to S^p\times S^q$. By 7.3 and 7.4, the diffeomorphism η satisfies conditions (d_1), (d_2), (d'_3), and (d_4). Let $\bar{L}=\eta\bullet\sigma_M(K)$ be the diff-spin of K. Applying 6.3 and 6.5 to this situation, we have the following theorem.

8.2 **Theorem.** *The $(m-1)$-simple fibred $(p+q+2m-1)$-knots L and $\bar{L}$ have isometric homotopy Seifert pairings, but if c and d are coprime, then the exteriors of L and $\bar{L}$ are not homotopy equivalent.*

We now spin the knots of 8.2 to obtain a more general result. For $t>0$, let $M^{(t)}=S^p\times S^q\times S^t\subsetneq S^{p+q+t+2m-1}$ with trivial framing, and let $\eta(t)=g(\lambda)\times \mathrm{id}_{S^t}:S^p\times S^q\times S^t\to S^p\times S^q\times S^t$. By 5.6, the diffeomorphism $\eta(t)$ satisfies conditions (d_1), (d_2), and (d'_3). Clearly, it also satisfies (d_4). Let $L^{(t)}=\sigma_{M^{(t)}}(K)$ and $\bar{L}^{(t)}=\eta(t)\ \sigma_{M^{(t)}}(K)$.

8.3 **Theorem.** *The $(m-1)$-simple fibred $(p+q+t+2m-1)$-knots $L^{(t)}$ and $\bar{L}^{(t)}$ have isometric homotopy Seifert pairings, but if c and d are coprime, then the exteriors of $L^{(t)}$ and $\bar{L}^{(t)}$ are not homotopy equivalent.*

Applying the computations of Sect. 7 to 8.2, we obtain the following examples.

8.4 *Example.* With $p=4$, $q=2$, $n=3$, $m=2$, and $\lambda:S^4\to SO(3)$ as in 7.5. Then $d=2$, so choose a 1-simple fibred 3-knot K whose monodromy has odd homotopy order c. We obtain 1-simple fibred 9-knots L and $\bar{L}$ having isometric homotopy Seifert pairings, but whose exteriors are not homotopy equivalent.

8.5 *Example.* With $p=10$, $q=7$, $n=5$, $m=3$, and $\lambda:S^{10}\to SO(8)$ as in 7.6. Then d is even, so choose a 2-simple fibred 5-knot K whose monodromy has homotopy order c coprime to d. We obtain 2-simple fibred 22-knots L and $\bar{L}$ having isometric homotopy Seifert pairings, but whose exteriors are not homotopy equivalent.

8.6 *Example.* With $p=10$, $q=4$, $n=11$, $m=6$, and $\lambda:S^{10}\to SO(8)$ as in 7.7. Then d is even, so choose a 5-simple fibred 11-knot K whose monodromy has homotopy order coprime to d. We obtain 5-simple fibred 25-knots L and $\bar{L}$ having isometric homotopy Seifert pairings, but whose exteriors are not homotopy equivalent.

We assemble all of the above into the following theorem.

8.7 **Theorem.** *For each integer $n\geqq 9$, there exist infinitely many pairs $(L_i,\bar{L}_i)$ of fibred n-knots such that*

(1) L_i and $\bar{L}_i$ have isometric homotopy Seifert pairings:

(2) the exterior of L_i is homotopically distinct from the exterior of L_j for $i\neq j$, and also from the exterior of $\bar{L}_j$ for all j;

(3) the knots L_i and $\bar{L}_i$ are 1-simple if $n\geqq 9$, 2-simple if $n\geqq 22$, and 5-simple if $n\geqq 25$.

Proof. Choose infinitely many pairs (a_i, b_i) such that a_i is coprime to b_i, and $a_i b_i \neq a_j b_j$ for $i \neq j$. For $n=9$ (resp. $n=22$; $n=25$), let L_i and $\bar{L}_i$ be the n-knots of 8.4 (resp. 8.5; 8.6), obtained from the 1-simple 3-knots (resp. 2-simple 22-knots; 5-simple 25-knots) K_{a_i, b_i}. [This imposes further conditions on the possible pairs (a_i, b_i), but we can still choose infinitely many of them.] For $9<n<22$ (resp. $22<n<25$; $n>25$), define L_i and $\bar{L}_i$ to be the knots $L_i^{(t)}$ and $\bar{L}_i^{(t)}$ of 8.3 with $t=n-9$ (resp. $t=n-22$; $t=n-25$). Then L_i and $\bar{L}_i$ are the desired knots. □

8.8. *Concluding remarks.* By the remarks after 7.4, the knots of 8.2 reside in the connectivity/dimension range

$$\frac{m-1}{p+q+2m-1} \leqq \frac{m-1}{2q+3m-1} \leqq \frac{m-1}{3(m+1)} < \frac{1}{3}.$$

We conjecture that for any positive number $s<1/3$, there exist integers n and r, with $r/n \geqq s$, and r-simple fibred n-knots L and $\bar{L}$, having isometric homotopy Seifert pairings and non-homotopy equivalent exteriors. Theorem 8.7 verifies this conjecture for $s=1/5$.

References

[C] Cappell, S.E.: Superspinning and knot complements. Topology of manifolds (Proc. Inst. Univ. of Georgia, Athens, GA, 1969), pp. 358–383. Chicago: Markham 1970

[F] Farber, M.S.: Isotopy types of knots of codimension two. Trans. Am. Math. Soc. **261**, 185–209 (1980)

[H] Hacon, D.: Iterated torus knots. Math. Proc. Camb. Phil. Soc. **80**, 57–60 (1976)

[K] Kearton, C.: Blanchfield duality and simple knots. Trans. Am. Math. Soc. **202**, 141–160 (1975)

[Ke] Kervaire, M.: Some nonstable homotopy groups of Lie groups. Ill. J. Math. **4**, 161–169 (1960)

[Kl] Klein, J.: Knots in the stable range: an extension of Farber's classification theorem. Preprint (1990)

[L] Levine, J.: An algebraic classification of some knots of codimension two. Comment. Math. Helv. **45**, 185–198 (1970)

[O] Ōguchi, K.: 2-Primary components of the homotopy groups of some Lie groups. Proc. Japan Acad. **38**, 235–238 (1962)

[R] Roseman, D.: Spinning knots about submanifolds; spinning knots about projections of knots. Topology Appl. **31**, 225–241 (1989)

[S] Spanier, E.H.: Duality in homotopy theory. Mathematika **2**, 56–80 (1955)

[Su] Suciu, A.I.: Inequivalent frame-spun knots with the same complement. Preprint (1990)

[T] Toda, H.: Composition methods in the homotopy groups of spheres (Ann. Math. Stud., vol. 49). Princeton: Princeton University Press 1962

[Tr 1] Trotter, H.F.: On S-equivalence of Seifert matrices. Invent. Math. **20**, 173–207 (1973)

[Tr 2] Trotter, H.F.: Knot modules and Seifert matrices, in: Knot theory, (Lect. Notes Math., vol. 685, pp. 291–299). Berlin Heidelberg New York: Springer 1978

[W] Wall, C.T.C.: Classification problems in differential topology. II. Diffeomorphisms of handlebodies. Topology **2**, 263–272 (1963)

[Wh] Whitehead, G.: Elements of homotopy theory (Graduate Texts Math., vol. 61). Berlin Heidelberg New York: Springer 1978