POINCARÉ DUALITY EMBEDDINGS AND FIBREWISE HOMOTOPY THEORY, II

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Abstract

We prove a relative Poincaré embedding theorem for maps of pairs into a Poincaré pair. This result has applications: it is the engine of the *compression theorem* of the author's paper in *Algebr. Geom. Topol.* **2**, (2002) 311–336, it yields a Poincaré space version of Hudson's embedding theorem, and it can be used to equip 2-connected Poincaré spaces with handle decompositions.

1. Introduction

In [**9**], the author proved an embedding theorem for a finite complex equipped with a map to a Poincaré complex. The aim of the present work will be to *relativize* this embedding result and afterwards provide applications of it.

Here is the context in which we place ourselves: let (K, L) be a cofibration pair such that K and L are homotopy finite. Let $(X, \partial X)$ be a Poincaré space of dimension n. Suppose that

$$f := (f_K, f_L) \colon (K, L) \to (X, \partial X)$$

is a map such that $f_L \colon L \to \partial X$ is the underlying map of a specified Poincaré embedding (of the kind that was called a *PD embedding* in [**9**]). Then we ask the following.

QUESTION 1 *Can one find an extension of the given embedding of L in ∂X to a relative Poincaré embedding K in X with underlying map f?*

Roughly, a *relative Poincaré embedding* of (K, L) in $(X, \partial X)$ is a decomposition of $(X, \partial X)$ up to homotopy as

$$(K, L) \cup_{(A_K, A_L)} (C_K, C_L),$$

such that

- K inherits the structure of an n-dimensional Poincaré space with boundary $L \cup_{A_L} A_K$ (A_K is called the *gluing space*, or *normal data*),
- C_K (the *complement* of K) is a Poincaré space with boundary $A_K \cup_{A_L} C_L$,
- L inherits the structure of a Poincaré space with boundary A_L, and
- C_L is a Poincaré space with boundary A_L.

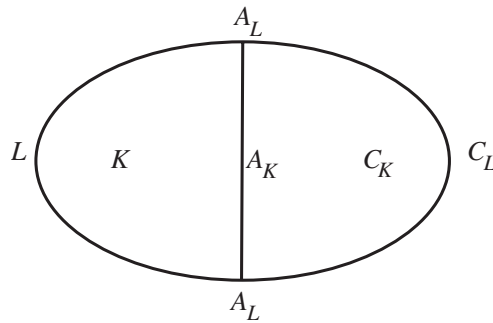---

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Fig. 1

In other words, a relative Poincaré embedding is a certain kind of 'Poincaré stratification' of the pair $(X, \partial X)$ having two strata (the codimension zero stratum given by $(K \sqcup C_K, L \sqcup C_L)$ and the codimension one stratum given by (A_K, A_L)). For the exact definition, see 2.2 below. The decomposition of $(X, \partial X)$ is depicted in Fig. 1.

If K is obtained from L up to homotopy by attaching cells of dimension at most k, we say that the *relative dimension* of (K, L) is at most k and we write $\dim(K, L) \leqslant k$. The main result of this paper is that Question 1 has an affirmative answer when the map f_K has sufficient connectivity.

THEOREM A *Suppose that $f_K : K \to X$ is r-connected. Then f Poincaré embeds, relative to the given embedding of L in ∂X, provided $k \leqslant n-3$ and $r \geqslant 2k - n + 2$.*

REMARK 1.1 This is the Poincaré version of a result of Hodgson giving criteria for a finite CW pair to embed up to homotopy in a PL manifold with boundary [**4**] (actually, our Poincaré version suffers from a loss of one dimension in the connectivity estimate[†]). When X is a PL manifold we can apply the Browder–Casson–Sullivan–Wall theorem [**17**, Chapter 11] to recover Hodgson's manifold result.

REMARK 1.2 A special case occurs when K and X are highly connected: in this instance any map f_K will be highly connected, and will therefore Poincaré embed relative to L. Thus we get a Poincaré version of a result of Irwin [**6**].

Overview of the proof

As in [**9**], our proof will be homotopy theoretic: we begin by constructing a relative embedding in stable codimension, by replacing X with $X \times D^j$. By downward induction on codimension, it is then enough to consider the case $j=1$.

Let (W_K, W_L) be the complement of a relative Poincaré embedding of the composite

$$(K, L) \xrightarrow{f} (X, \partial X) \subset (X \times I, \partial(X \times I)).$$

[†] Whether the dimension loss phenomenon in the Poincaré case is real or merely a defect of the technique of proof is currently unknown. The dimension loss was already observed in certain cases by Levitt [**12**, p. 402]. However, in the metastable range, when $L = \emptyset$ one can recover the lost dimension (see [**8**, Theorem F]).

Then W_L is automatically a fibrewise suspension of a space over ∂X. Moreover, the connectivity assumption allows us to apply [**9**, Theorem 4.7] to fibrewise desuspend W_K over X; any choice of fibrewise desuspension is then a candidate for the complement of a relative embedding of f.

We then need to find the glue which binds (K, L) with the chosen candidate for the complement. This is obtained by applying the dual Blakers–Massey theorem for 3-cubes [**2**] and truncation techniques developed in [**9**].

Applications

Theorem A was first announced in [**8**] and was the crux of the proof of the *compression theorem* (the main result of that paper [**8**, Theorem A]). Furthermore, the compression theorem was shown to have a variety of applications, including

- embeddings of spheres in Poincaré spaces [**8**, Theorem E],[‡]

- a Levine style Poincaré embedding theorem [**8**, Theorem D],

- the existence of diagonal (that is, tangential) Poincaré embeddings [**8**, Corollary H], and

- a sharpening of the inequality of the main theorem of [**9**] by one dimension, with the additional assumption that one is working in the metastable range [**8**, Theorem F] (cf. Remark 1).

We now give four additional applications of Theorem A.

Application 1: *concordance*

Let e_0 and e_1 be Poincaré embeddings with underlying maps $f_0, f_1\colon K \to X$, and suppose that we are given a homotopy $F\colon K \times I \to X$ from f_0 to f_1. In particular, we have an associated embedding with underlying map $f_0 \sqcup f_1\colon K \times \{0, 1\} \to \partial(X \times [0, 1])$; we donote this embedding by e. Consider the associated map of pairs

$$F\colon (K \times [0, 1], K \times \{0, 1\}) \to (X \times [0, 1], \partial(X \times [0, 1])) .$$

DEFINITION 1.3 A *concordance* from e_0 to e_1 is a Poincaré embedding of F relative to e.

The following is a fundamental uniqueness result for Poincaré embeddings; it complements the main result of [**9**].

COROLLARY B *Suppose e_0, e_1 and F are as above. Assume K is homotopy equivalent to a* CW *complex of dimension at most k, and that $(X, \partial X)$ is a Poincaré space of dimension n. Let r denote the connectivity of f_0. Then e_0 and e_1 are concordant provided that $k \leqslant n-3$ and $r \geqslant 2k - n + 3$.*

The result follows by applying Theorem A to the pair (F, e).

Application 2: *Poincaré embeddings of disks*

Let

$$f = (f_{D^k}, f_{S^{k-1}})\colon (D^k, S^{k-1}) \to (X, \partial X)$$

be a map, with $(X, \partial X)$ a Poincaré space of dimension n. (Note that in what follows we do not *a priori* assume that $f_{S^{k-1}}\colon S^{k-1} \to \partial X$ is the underlying map of a Poincaré embedding.)

[‡] See also [**10**]; this is a key ingredient in the approach to Poincaré surgery promulgated by Bill Richter and the author.

THEOREM C *Assume that $(X, \partial X)$ is $(2k-n+2)$-connected, X is 2-connected, and $k \leqslant n-3$. Then f relatively Poincaré embeds.*

REMARK 1.4 The PL version of Theorem C (due to Hudson [5]) is true without the assumption that X is 2-connected. Furthermore, $(X, \partial X)$ is only required to be $(2k-n+1)$-connected in the PL case. I suspect that Theorem C is actually true without the additional hypotheses.

Application 3: *Poincaré cobordisms*

By a *Poincaré cobordism*, we mean a Poincaré space W whose boundary has the form $\partial W = \partial_0 W \amalg \partial_1 W$.

DEFINITION 1.5 A Poincaré cobordism is said to be *elementary of index k* (*relative to $\partial_0 W$*) if

(1) W is, up to homotopy, obtained by attaching a single cell of dimension k to $\partial_0 W$, that is, the inclusion $\partial_0 W \to W$ factors as

$$\partial_0 W \to (\partial_0 W) \cup D^k \overset{\simeq}{\to} W \,.$$

(2) Similarly, W is obtained from $\partial_1 W$ by attaching an $(n-k)$-cell, where n is the dimension of $(W, \partial W)$.

REMARK 1.6 When $3 \leqslant k \leqslant n-3$, conditions (1) and (2) are equivalent (by [14]) to conditions (1) and

(2)$'$ the pair $(W, \partial_1 W)$ is 2-connected.

EXAMPLE 1.7 One way to obtain an elementary Poincaré cobordism from a closed Poincaré space $\partial_0 W$ of dimension $n-1$ is to take the trace of a surgery on a framed Poincaré embedded sphere in $\partial_0 W$: start with a *codimension zero Poincaré embedding* $e \colon S^{k-1} \times D^{n-k} \subset \partial_0 W$ with complement C. This is just a Poincaré embedding diagram of the form

$$
\begin{array}{ccc}
S^{k-1} \times S^{n-k-1} & \longrightarrow & C \\
\downarrow & & \downarrow \\
S^{k-1} \times D^{n-k} & \longrightarrow & \partial_0 W \,,
\end{array}
$$

that is, the gluing space of the embedding is specified to be $S^{k-1} \times S^{n-k-1}$.

Define

$$W := (D^k \times D^{n-k} \times 1) \ \cup_e \ (\partial_0 W \times [0, 1]).$$

Then W is an elementary Poincaré cobordism of index k with

$$\partial_1 W := D^k \times S^{n-k-1} \cup_{S^{k-1} \times S^{n-k-1}} C.$$

Thus $\partial_1 W$ is the effect of doing surgery on the framed embedding e, and W is the trace of the surgery. Note that, by turning W 'upside down', one can view W as the trace of a surgery of index $n-k$ relative to $\partial_1 W$ using the evident codimension zero Poincaré embedding $D^k \times S^{n-k-1} \to \partial_1 W$.

QUESTION 2 *Does every elementary Poincaré cobordism W arise as the trace of a Poincaré surgery?*

The following result gives a partial answer.

THEOREM D *Let W^n be an elementary Poincaré cobordism of index k. Assume that W is 2-connected and $3 \leqslant k \leqslant n-3$. Then up to homotopy, W is the trace of a surgery on a suitable codimension zero Poincaré embedding $e\colon S^{k-1} \times D^{n-k} \subset \partial_0 W$.*

REMARK 1.8 This result is originally due to Bill Richter (unpublished), using methods different from those developed here. In fact, Richter only requires W to be 1-connected.

For applications, it is useful to consider the more general situation in which a cobordism is given by attaching more than one k-cell.

DEFINITION 1.9 A Poincaré cobordism W is said to be of *index k relative to $\partial_0 W$* if W is obtained up to homotopy from $\partial_0 W$ by attaching a finite set of k-cells (sequentially), and similarly, W is also obtained from $\partial_1 W$ by attaching a finite set of $(n-k)$-cells.

THEOREM E *Assume that W is 2-connected and $3 \leqslant k \leqslant n-3$. If W is a Poincaré cobordism of index k, then W is the trace of a finite sequence of surgeries on suitable codimension zero Poincaré embeddings $e\colon S^{k-1} \times D^{n-k} \subset \partial_0 W$ (that is, W is the result of performing a finite collection of handle attachments to $\partial_0 W$, each having index k).*

Application 4: *Poincaré handlebodies*

Let X be a connected Poincaré duality space of dimension n, with $\partial X = \emptyset$. A *handle decomposition* for X consists of a filtration

$$X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$$

satisfying

- $X_{-1} = \emptyset$;
- each X_i is a Poincaré duality space of dimension n;
- the inclusion $X_i \subset X_{i+1}$ is such that X_{i+1} is obtained from X_i by attaching a finite number of Poincaré handles of index $i+1$ to ∂X_i.

THEOREM F *Let X be a 2-connected closed Poincaré space of dimension n. Then X admits a handle decomposition.*

REMARK 1.10 A statement of this kind recalls the early work of Levitt [**11**]. Handle decompositions are essentially a kind of 'patch space' in the sense of Jones [**7**]. Handle decompositions without the connectivity assumption on X are deduced in the book of Hausmann and Vogel [**3**] (who use manifold techniques as their main tool). Richter (unpublished) was the first to deduce handle decompositions via a homotopy theoretic assault. Richter's technique is different from mine (his main tool is Ganea theory), and he only requires X to be 1-connected.

Note to the reader. This paper is written for those who already have some familiarity with the first paper in this series [**9**].

OUTLINE 1 Section 2 is, for the most part, preliminary material. Aside from giving a rigorous definition of relative Poincaré embeddings, we also define the stabilization construction. In Section 3 we show that every map relatively Poincaré embeds upon decompression into large codimension. Section 4 contains the proof of Theorem A. In Section 5 we prove Theorems D and E. In Section 6 we prove Theorem F. In Section 7 we prove Theorem C.

2. Preliminaries

This section is not intended to be complete. See [**9**] for a more detailed discussion of this material.

Spaces

As in [**9**], we work in the Quillen model category **Top** of compactly generated topological spaces [**13**]. Recall that the weak equivalences of **Top** are the weak homotopy equivalences. For the most part, we work with cofibrant spaces: that is, those spaces which are (retracts of) spaces built up from the empty space by attaching cells (if the result of a construction destroys cofibrancy, we typically apply functorial cofibrant replacement). Knowledge of homotopy limits and colimits is assumed on the part of the reader (see [**1**]).

A pair (Y, A) satisfies $\dim(Y, A) \leqslant k$ if there exists a factorization

$$A \to Z \xrightarrow{\sim} Y$$

such that Z is built up from A by attaching cells of dimension $\leqslant k$ (that is, (Y, A) is weak equivalent rel A to a relative CW complex of relative dimension at most k). As a special case, we write $\dim Y \leqslant k$ when $A = \emptyset$.

A commutative square of spaces

$$
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow & & \downarrow \\
B & \longrightarrow & D
\end{array}
$$

is *j-coCartesian* if the induced map $B \times 0 \cup A \times [0, 1] \cup C \times 1 \to D$ is j-connected. Similarly, it is *j-Cartesian* if the map from A into the homotopy pullback of $B \to D \leftarrow C$ is j-connected.

Given a map of spaces $f \colon Y \to Z$, we will sometimes let $(\bar{Z}, Y)$ denote the mapping cylinder pair $(Z \cup_{f \times 0} (Y \times [0, 1]), Y \times 1)$.

Fibrewise spaces

For a map of spaces $A \to X$, we consider the category

$$A \backslash \mathbf{Top} / X$$

whose objects are spaces Y equipped with a choice of factorization $A \to Y \to X$. A morphism $Y \to Y'$ is a map of underlying spaces which is compatible with the structure maps. For example, when A is the empty space we obtain $\mathbf{Top}/X$, which is the category of spaces over X. It was proved in [**9**, 4.4] that $A \backslash \mathbf{Top} / X$ is a model category: the *weak equivalences* are morphisms whose underlying maps of spaces are weak homotopy equivalences.

Fibrewise suspension is the functor

$$A \backslash \mathbf{Top} / X \to (\Sigma_X A) \backslash \mathbf{Top} / X$$

given by mapping Y to $\Sigma_X Y := X \times 0 \cup Y \times [0, 1] \cup Y \times 1$. One of the important tools used to construct embedded Poincaré thickenings is the *fibrewise desuspension theorem* [**9**, 4.7, 4.9] which gives sufficient criteria for an object to be a fibrewise suspension up to weak equivalence.

Poincaré duality spaces

We use Wall's definition [**16**]: let $(X, \partial X)$ be a homotopy finite pair such that X is equipped with a local coefficient system $\mathcal{L}$ which is pointwise free abelian of rank one. One says that X is a *Poincaré space of (formal) dimension n* if there exists a class $[X] \in H_n(X, \partial X; \mathcal{L})$ (called a *fundamental class*) such that

- the cap product homomorphism

$$\cap [X] \colon H^*(X; \mathcal{M}) \overset{\cong}{\to} H_{n-*}(X, \partial X; \mathcal{L} \otimes \mathcal{M})$$

  is an isomorphism in all degrees for all local coefficient systems $\mathcal{M}$ on X.

- Moreover, if $[\partial X] \in H_{n-1}(\partial X; \mathcal{L}_{|\partial X})$ denotes the image of $[X]$ with respect to the boundary homomorphism in the homology long exact sequence of $(X, \partial X)$, then the cap product homomorphism

$$\cap [\partial X] \colon H^*(\partial X; \mathcal{M}) \overset{\cong}{\to} H_{n-1-*}(\partial X, \mathcal{L}_{|\partial X} \otimes \mathcal{M})$$

  is an isomorphism in all degrees for all local coefficient systems $\mathcal{M}$ on ∂X.

In particular, ∂X is itself a Poincaré space (with empty boundary).

Codimension zero Poincaré embeddings

If $f \colon M^n \to X^n$ is a map in which M and X are Poincaré duality spaces of dimension n (here M and X are allowed to have boundaries), then a codimension zero *Poincaré embedding* of f consists of an n-dimensional Poincaré space C with boundary $\partial M \sqcup \partial X$ such that f and C assemble to give a homotopy decomposition of X:

$$X \simeq M \cup_{\partial M} C$$

in such a way that the fundamental class of X is compatible with those of M and C in the above amalgamation. The space C is called the *complement* of the embedding. For more details, see [**8**].

Embedded thickenings

A homotopy finite space K might admit more than than one Poincaré boundary. If $(K', \partial K')$ is a Poincaré space with K' having the homotopy type of K, and K' is codimension zero embedded in a Poincaré space X, then it is reasonable to think of K' as the analogue of a regular neighbourhood of K in X. This idea gives rise to the notion of embedded Poincaré thickening.

DEFINITION 2.1 Let X be an n-dimensional Poincaré space, let K be a homotopy finite space, and let $f \colon K \to X$ be a map. An *embedded thickening* of f consists of

- (*Stratification*) a commutative coCartesian square of spaces

$$
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow & & \downarrow{\scriptstyle j} \\
K & \underset{f}{\longrightarrow} & X
\end{array}
$$

  and a factorization of the inclusion $\partial X \to X$ as $\partial X \to C \overset{j}{\to} X$. We also demand that A and C are homotopy finite. Furthermore, these data are assumed to satisfy

- (*Poincaré duality*). We require that the image of a fundamental class for X under the composite $H_n(X, \partial X) \to H_n(X, C) \cong H_n(\bar{K}, A)$ provides $(\bar{K}, A)$ with the structure of a Poincaré space (twisted coefficients are implicit here; we remind the reader that $\bar{K}$ is the mapping cylinder of $A \to K$). Similarly, the image of a fundamental class of $[X]$ under the composite $H_n(X, \partial X) \to H_n(X, K \amalg \partial X) \cong H_n(\bar{C}, A \amalg \partial X)$ provides $(\bar{C}, A \amalg \partial X)$ with the structure of a Poincaré space.

- (*Weak transversality*). If $\dim K \leqslant k$, then $A \to K$ is $(n-k-1)$-connected.

The space C is the *complement*, and A is the *gluing space* of the embedded thickening. If $\dim K \leqslant k$, then we say that the (*homotopy*) *codimension* of the embedded thickening is at least $n-k$. The weak transversality condition is supposed to mirror the fact that a regular neighbourhood of a k-dimensional polyhedron in an n-dimensional manifold has this property.

TERMINOLOGY In [**9**] we used 'PD embedding' to refer to embedded thickenings. Beginning in [**8**], we decided to change the name to emphasize the difference between embedded thickenings and codimension zero Poincaré embeddings. In light of these distinctions, note that the 'embeddings' referred to in the Introduction are meant to be embedded thickenings.

Relative embedded thickenings

We now relativize the foregoing. Let (K, L) be a cofibration pair such that K and L are homotopy finite. Let X be an n-dimensional Poincaré space. Fix a map

$$f = (f_K, f_L) \colon (K, L) \to (X, \partial X).$$

DEFINITION 2.2 A *embedded thickening* of f consists of a commutative diagram of cofibration pairs of homotopy finite spaces

$$
\begin{array}{ccc}
(A_K, A_L) & \longrightarrow & (C_K, C_L) \\
\downarrow & & \downarrow \\
(K, L) & \xrightarrow{\;\;f\;\;} & (X, \partial X)
\end{array}
$$

such that

- (*Stratification*). Each of the associated diagrams of spaces

$$
\begin{array}{ccc}
A_K & \longrightarrow & C_K \\
\downarrow & & \downarrow \\
K & \xrightarrow{f_K} & X
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
A_L & \longrightarrow & C_L \\
\downarrow & & \downarrow \\
L & \xrightarrow{f_L} & \partial X
\end{array}
$$

is coCartesian and the latter of these diagrams is an embedded thickening of L in ∂X;

- (*Duality*). The image of the fundamental class of X with respect to the composite

$$H_n(X, \partial X) \to H_n(\bar{X}, \partial X \cup_{C_L} C_K) \cong H_n(\bar{K}, L \cup_{A_L} A_K)$$

gives $(\bar{K}, L \cup_{A_L} A_K)$ the structure of an n-dimensional Poincaré space (here, coefficients are given by pulling back the local system for X). Similarly, $(\bar{C}_K, C_L \cup_{A_L} A_K)$ has the structure of a Poincaré space with fundamental class induced from X.

• (*Weak transversality*). If $\dim(K, L) \leqslant k$, then the map $A_K \to K$ is $(n-k-1)$-connected.

If an embedded thickening e of f_L is *a priori* specified, then an embedded thickening of f which on L coincides with e is said to be an *embedded thickening of f relative to e*. Another way to formulate the latter is as follows: notice that e determines a map $C_L \cup_{A_L} A_K \to X$, and an embedded thickening of f relative to e amounts to choosing an object

$$C_K \in (C_L \cup_{A_L} A_K)\backslash \mathbf{Top}/X$$

which satisfies the stratification, duality and weak transversality axioms.

Stabilization

Let $(X, \partial X)$ be an n-dimensional Poincaré space. Let

$$\xi \colon S(\xi) \to X$$

be a spherical fibration with fibre S^{j-1}, and let $D(\xi)$ denote the mapping cylinder of ξ. If $Z \to X$ is a map, then $\xi|Z$ will denote the pullback of ξ to Z.

DEFINITION 2.3 (twisted fibrewise suspension) For a map $Z \to X$, set

$$\Sigma^\xi Z = \text{hocolim}\,(Z \leftarrow S(\xi|Z) \to S(\xi))\,.$$

As a special case, when ξ is the trivial fibration with fibre S^0, we have an identification of $\Sigma^\xi Z$ with $\Sigma_X Z$, the fibrewise suspension of $Z \to X$.

When $Z = \partial X$ we define

$$\partial D(\xi) := \Sigma^\xi \partial X\,.$$

If we identify $D(\xi)$ with $\text{hocolim}\,(\partial X \overset{\sim}{\leftarrow} D(\xi|\partial X) \to D(\xi))$, then $(D(\xi), \partial D(\xi))$ is a Poincaré space of dimension $n+j$.

Now, suppose we are given relative embedded thickening of f with associated diagram

$$\begin{array}{ccc}
(A_K, A_L) & \longrightarrow & (C_K, C_L) \\
\downarrow & & \downarrow \\
(K, L) & \xrightarrow{\;\;f\;\;} & (X, \partial X).
\end{array}$$

Using ξ and its restriction along f_K and f_L, we may apply the twisted suspension construction to obtain a commutative diagram

$$\begin{array}{ccc}
(\Sigma^{\xi|K} A_K, \Sigma^{\xi|L} A_L) & \longrightarrow & (\Sigma^\xi C_K, \Sigma^\xi C_L) \\
\downarrow & & \downarrow \\
(D(\xi|K), D(\xi|L)) & \longrightarrow & (D(\xi), \partial D(\xi))
\end{array}$$

which defines a relative embedded thickening of $(D(\xi|K), D(\xi|L))$ in $(D(\xi), \partial D(\xi))$. Using the evident homotopy equivalence

$$(K, L) \overset{\simeq}{\to} (D(\xi|K), D(\xi|L))$$

(and taking a suitable mapping cylinder) we obtain a relative embedded thickening of the composite

$$(K, L) \xrightarrow{\simeq} (D(\xi|K), D(\xi|L)) \to (D(\xi), \partial D(\xi)).$$

A special case of this construction occurs when $S(\xi) \to X$ is the trivial fibration with fibre S^{j-1}. If this is the case, the pair $(D(\xi), \partial D(\xi))$ is identified with $(X \times D^j, \partial(X \times D^j))$, and the new embedded thickening is called the j-fold *decompression*.

3. Relative embedded thickenings in stable codimension

Assume that $f = (f_K, f_L): (K, L) \to (X, \partial X)$ is such that $f_L: L \to \partial X$ is the underlying map of an embedded thickening e. We want to prove that a suitable iterated decompression of e extends to a relative embedded thickening of f.

   The argument given here has the advantage of being relatively short.   However, its main disadvantage is that it is not manifold-free (transversality and the existence of regular neighbourhoods are needed). In another paper, we will show how to obtain relative embeddings using equivariant Spanier–Whitehead duality (with respect to a topological group model for the loop space of X).

   We begin with a special case.

LEMMA 3.1 *Assume* $\dim(K, L) \leqslant k$. *If* $(X, \partial X)$ *has the homotopy type of a compact PL manifold of dimension* $n \geqslant 6$, *and* $n \geqslant 2k+1$, *then* f *embedded thickens relative to* e.

Proof. Without loss in generality, we can assume that X is a compact PL manifold. Consider the restriction $f_L: L \to \partial X$ together with its given embedded thickening. The Browder–Casson–Sullivan–Wall theorem [17, Chapter 11] shows that this embedded thickening can be *linearized:* that is, it is concordant to an embedded PL thickening of $f_L: L \to \partial X$ (recall that an *embedded PL thickening*[§] of $f_K: K \to X$ in the sense of [15] consists of a PL codimension one splitting $\partial X = V \cup_{\partial V} C$ and a homotopy equivalence $h: K \xrightarrow{\simeq} V$ such that h followed by the inclusion $V \subset X$ is homotopic to f). Thus, one can assume without loss in generality that L is a compact codimension zero PL submanifold of ∂X. The proof is now completed by applying a straightforward cell-by-cell induction using general position and the existence of regular neighbourhoods to extend the submanifold $L \subset \partial X$ to a relative embedded PL thickening of f.

   Let $f: (K, L) \to (X, \partial X)$ be a map such that f_L comes equipped with an embedded thickening e. Then for any integer $j \geqslant 0$, e determines an embedded thickening e_j of the composite map

$$L \xrightarrow{e} \partial X \subset \partial(X \times D^j)$$

(this uses j-fold decompression and the inclusion $(\partial X) \times D^j \subset (X \times D^j)$). We are now in a position to handle the general case.

THEOREM 3.2 *There exists an integer* $j \gg 0$ *such that the composite*

$$(K, L) \xrightarrow{f} (X, \partial X) \xrightarrow{\subseteq} (X \times D^j, \partial(X \times D^j))$$

embedded thickens relative to e_j.

[§] This is sometimes called an *embedding up to homotopy*.

Proof. Let $\nu\colon S(\nu) \to X$ denote the Spivak normal fibration, and let $D(\nu)$ denote its mapping cylinder. then $D(\nu)$ is a Poincaré space with boundary $S(\nu) \cup_{S(\nu_{|\partial X})} \partial X$. Moreover, $(D(\nu), S(\nu))$ has the homotopy type of a compact manifold.

Consequently, by 3.1 the map

$$(K, L) \xrightarrow{f} (X, \partial X) \subset (D(\nu), \partial D(\nu))$$

relative embedded thickens. Call this relative embedded thickening e'. The proof is finished by applying the stabilization construction to e' (cf. section 2) using the Spivak *tangent* fibration of X.

DEFINITION 3.3 *Stable concordance* is the equivalence relation generated by concordance and decompression. Specifically, suppose one has embedded thickenings e_0 and e_1 with underlying maps $f_0, f_1\colon K \to X$, and a homotopy $F\colon K \times I \to X$ from f_0 to f_1. Then e_0 and e_1 are *stably concordant* (with respect to F) if they are concordant after taking a suitable iterated decompression of e_0 and e_1.

COROLLARY 3.4 *Let* $f\colon K \to X$ *be a map which embedded thickens. Then there is exactly one stable concordance class of embedded thickening having* f *as underlying map* (*with respect to the constant homotopy of* f).

4. Proof of Theorem A

The argument is along the same lines as the proof of the main theorem of [**9**]. For this reason, we present the proof with less formality. For more details, we refer the reader to [**9**, section 6].

Assume that $f_L\colon L \to \partial X$ comes equipped with an embedded thickening e, whose diagram is denoted by

$$
\begin{array}{ccc}
A_L & \longrightarrow & C_L \\
\downarrow & & \downarrow \\
L & \underset{f_L}{\longrightarrow} & \partial X\,.
\end{array}
$$

In what follows, r will denote the connectivity of $f\colon K \to X$. Assume that $r \geqslant 2k-n+2$ and $k \leqslant n-3$, where k denotes the relative dimension of (K, L) and n denotes the dimension of $(X, \partial X)$. We wish to show that the embedded thickening of f_L extends to a relative embedded thickening of f.

Step 1. By 3.2 and downward induction on codimension, we may assume that the composite

$$(K, L) \xrightarrow{f} (X, \partial X) \to (X \times I, \partial(X \times I))$$

is the underlying map of an embedded thickening which restricts to the given one on L. It is sufficient to show that this relative embedded thickening compresses into $(X, \partial X)$. The next step is to construct a candidate for the complement.

Step 2. Let us assume the relative embedded thickening of step 1 is given by

$$
\begin{array}{ccc}
(A'_K, A'_L) & \longrightarrow & (W_K, W_L) \\
\downarrow & & \downarrow \\
(K, L) & \underset{f'}{\longrightarrow} & (X \times I, \partial(X \times I))
\end{array}
$$

where the map f' is given by f followed by the inclusion $(X, \partial X) \subset (X \times I, \partial(X \times I))$. Since the diagram restricts along L to the diagram of decompression of e, there is a Poincaré space (C_L, A_L) and an identification

$$(W_L, A_L') \;\simeq\; (\Sigma_X C_L, \Sigma_L A_L).$$

PROPOSITION 4.1 *With respect to the assumptions of Theorem* A, W_K *fibrewise desuspends over* X *relative to* $W_L = \Sigma_X C_L$. *That is, there exists an object* $C_K \in C_L \backslash \mathbf{Top}/X$ *and a weak equivalence*

$$\Sigma_X C_K \;\simeq\; W_K$$

in $(\Sigma_X C_L) \backslash \mathbf{Top}/X$.

Proof. The map $W_K \to X$ is $(n{-}k)$-connected, since it opposes the $(n{-}k)$-connected map $A_K' \to K$ in a homotopy coCartesian square. Furthermore, the map $\Sigma_X C_L \to W_K$ has the property that its (relative) cohomology (with any local coefficients) vanishes above degree $n{-}r$. This last assertion is a consequence of Poincaré duality: recall that ∂W_K is identified with $\Sigma_X C_L \cup_{\Sigma_L A_L} A_K'$, so the relative cohomology of $\Sigma_X C_L \to W_K$ is Poincaré dual to the relative homology of $A_K' \to W_K$. The latter, by excision, coincides with the relative homology of f_K. The proof is finished by applying the fibrewise desuspension theorem [**9**, Theorem 4.7].

Step 3. Consider the object $A_K' \in A_L' \backslash \mathbf{Top}/K$. Since $A_L' = \Sigma_L A_L$, we have a lifting problem,

$$
\begin{array}{ccccc}
\Sigma_L A_L & \longrightarrow & A_K' & \longrightarrow & W_K \\
\big\downarrow & \nearrow \ell' & \big\downarrow & \diagup \ell & \big\downarrow \\
\Sigma_K A_L & \longrightarrow & K & \longrightarrow & X .
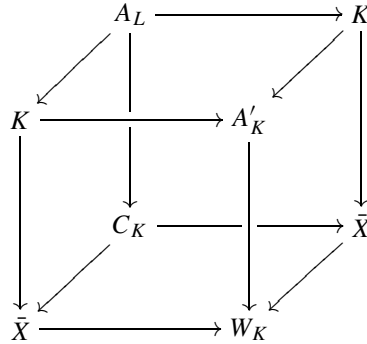\end{array}
$$

Note there is a preferred lift $\ell \colon \Sigma_K A_L \to W_K$ making the diagram homotopy commute (since the Poincaré boundary of W_K is identified with $\Sigma_X C_L \cup_{\Sigma_L A_L} A_K'$, the inclusion $\Sigma_X C_L \to W_K$ restricted to $\Sigma_K A_L$ gives this lift). We would like to back up ℓ to another lift ℓ' as indicated by the diagram.

Note that the right-hand square is $(r{+}n{-}k{-}1)$-Cartesian by the Blakers–Massey theorem [**2**]. Since $\dim(\Sigma_K A_L, \Sigma_L A_L) \leqslant k$ our hypotheses guarantee a solution to the lifting problem. Choose once and for all a lift

$$\ell' \colon \Sigma_K A_L \to A_K'$$

making the diagram homotopy commute. Replacing A_K' by a suitable mapping cylinder if necessary, we can arrange the lift so that the diagram strictly commutes. Assume that this has been done.

Step 4. The lift ℓ' can be displayed with the other data as a commutative 3-cube of spaces

$$
\begin{array}{ccc}
A_L & \longrightarrow & K \\
& & \\
K & \longrightarrow & A'_K \\
& & \\
C_K & \longrightarrow & \bar{X} \\
& & \\
\bar{X} & \longrightarrow & W_K
\end{array}
$$

where $\bar{X}$ denotes the mapping cylinder of the map $C_K \to X$ (the lift ℓ' amounts to the information contained in the top face of the cube).

Let B be the homotopy inverse limit of the punctured cube given by deleting A_L from the above 3-cube. In particular, we have a map

$$A_L \to B \,.$$

If we replace A_L by B in the above cube, the resulting 3-cube is commutative up to preferred homotopy. As in the discussion prior to [**9**, Claim 6.3], one can map this 3-cube to a strictly commuting one by a pointwise weak equivalence. To avoid notational clutter, we assume this has been done, but keep the present notation for the cube.

The following result is proved in the same way as [**9**, Claim 6.3].

LEMMA 4.2 *The square*

$$
\begin{array}{ccc}
B & \longrightarrow & K \\
\downarrow & & \downarrow \\
K & \longrightarrow & A'_K
\end{array}
$$

is $(2(n-k-1)+r)$-coCartesian. Morever, B is a connected space.

Step 5. We now apply the *coCartesian replacement theorem* [**9**, Theorem 4.2, Addendum 4.3] to the square of Lemma 4.2 and the map $A_L \to B$.

Since the relative cohomology of the map

$$\Sigma_K A_L \to A'_K$$

(with any local coefficient bundle) vanishes in degrees greater than or equal to n, we are entitled to apply [**9**, 4.3] (this is where the inequality $r \geqslant 2k - n + 2$ is used; for more details, see [**9**, 6.4]). This results in a space A_K and a factorization

$$A_L \to A_K \to B$$

such that the square

$$
\begin{array}{ccc}
A_K & \longrightarrow & K \\
\downarrow & & \downarrow \\
K & \longrightarrow & A'_K
\end{array}
$$

(given replacing B with A_K) is coCartesian.

The next lemma is proved in the same way as [**9**, Claim 6.5]

LEMMA 4.3 *The composite map $A_K \to B \to C_K$ together with the map $A_K \to K$ fits into a coCartesian square*

$$
\begin{array}{ccc}
A_K & \longrightarrow & C_K \\
\downarrow & & \downarrow \\
K & \longrightarrow & X.
\end{array}
$$

Finally, notice that what we really have is a coCartesian square of pairs

$$
\begin{array}{ccc}
(A_K, A_L) & \longrightarrow & (C_K, C_L) \\
\downarrow & & \downarrow \\
(K, L) & \longrightarrow & (X, \partial X).
\end{array}
$$

Finally, to see that this diagram is relative embedded thickening, follow the proof of [**9**, Claim 6.6] (with appropriate modifications). This completes the proof of Theorem A.

5. Proof of Theorems D and E

Theorem D is a special case of Theorem E. We first prove the former, and thereafter extend the proof to the latter.

Proof of Theorem D. Given an elementary Poincaré cobordism W^n of index k, relative to $\partial_0 W$ (where $\partial W = \partial_0 W \amalg \partial_1 W$), we wish to show that W is the trace of a Poincaré surgery on a codimension zero Poincaré embedding $S^{k-1} \times D^{n-k} \to \partial_0 W$, provided that $3 \leqslant k \leqslant n-3$ and W is 2-connected.

Case (1): $2k < n$. By [**9**, Theorem A] the map $S^{k-1} \to \partial_0 W$ admits an embedded thickening. Fixing this embedded thickening, we wish to extend it to a relative embedded thickening using the characteristic map $(D^k, S^{k-1}) \to (W, \partial W)$. One checks that the hypotheses of Theorem A hold in this case. The resulting relative embedded thickening of the characteristic map describes W as being obtained from $\partial_0 W$ by attaching a k-handle. This finishes case (1).

Case (2): $2k \geqslant n$. Turn W upside down and view it as an elementary cobordism of index $n-k$ relative to $\partial_1 W$.

The associated attaching map $S^{n-k-1} \to \partial_1 W$ admits an embedded thickening by [**9**, Theorem A]. We may now apply Theorem A to the characteristic map $(D^{n-k}, S^{n-k-1}) \to (W, \partial_1 W)$ to get a relative embedded thickening. This describes W as the trace of a surgery on a codimension zero Poincaré embedding $D^k \times S^{n-k-1} \to \partial_1 W$. Turning W rightside up, we see that it is is obtained from $\partial_0 W$ by attaching a k-handle.

Proof of Theorem E. The cobordism W is obtained from $\partial_0 W$ by sequentially attaching a finite set of k-cells.

First assume that $2k < n$. Then as in case (1) of the proof of Theorem D, we can represent the characteristic map $(D^k, S^{n-k-1}) \to (W, \partial_0 W)$ of the first cell by a relative embedded thickening. This allows us to rewrite $W = P \cup Q$ with P given by attaching a single k-handle to $\partial_0 P = \partial_0 Q$

and Q a cobordism of index k having one cell less (here the union is taken along $\partial_1 P = \partial_0 Q$; also we have deformed the attaching maps of the remaining cells of $(W, \partial_0 W)$ so that they are attached to $\partial_1 P$). Replacing W by Q and proceeding inductively, we obtain the desired decomposition of W.

In the case when $2k \geqslant n$, use case (2) in the proof of Theorem D and proceed *mutatis mudandis*.

6. Proof of Theorem F

In this section we prove that 2-connected closed Poincaré spaces admit Poincaré handle decompositions.

Let X^n be such a Poincaré space. If $n \leqslant 4$ then the fact that X is 2-connected implies by Poincaré duality that $X \simeq S^n$. The result clearly holds in this instance. So assume that $n \geqslant 5$. By a theorem of Wall [16], we may choose a CW decomposition of X of the form $X \simeq K \cup_\beta D^n$, where K is a 2-connected CW complex of dimension at most $n - 3$. Moreover, K can be chosen so that it has one 0-cell and no 1- or 2-cells.

We construct the filtration inductively. Suppose we have constructed a partial filtration

$$X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{k-1},$$

where $X_i \to X_{i+1}$ the underlying map of a Poincaré embedding. Furthermore, the $X_i \subset X$ are compatibly Poincaré embedded and each such map realizes the i-skeleton of X. Let C_{k-1} denote the complement of X_{k-1} in X. Then the diagram

$$
\begin{array}{ccc}
\partial X_{k-1} & \longrightarrow & C_{k-1} \\
\downarrow & & \downarrow \\
X_{k-1} & \longrightarrow & X
\end{array}
$$

is a homotopy pushout. Moreover, the relative Hurewicz theorem and excision gives an isomorphism

$$\pi_k(X, X_{k-1}) \cong \pi_k(C_{k-1}, \partial X_{k-1}).$$

Using this isomorphism, we see that the characteristic map $(D^k, S^{k-1}) \to (X_{k-1}, X)$ for each k-cell factors through $(C_{k-1}, \partial X_{k-1})$.

The set of k-cells of X therefore define a map

$$\alpha \colon (\partial X_{k-1} \ \cup \ (\coprod_r D^k), \partial X_{k-1}) \to (C_{k-1}, \partial X_{k-1}).$$

By Theorem A one can find a relative embedded thickening of α. We therefore have (1) a Poincaré cobordism V with $\partial_0 V = \partial X_{k-1}$, (2) a Poincaré embedding rel ∂X_{k-1} of V in C_{k-1} and (3) a homotopy equivalence

$$V \ \simeq \ \partial X_{k-1} \cup (\coprod_r D^k) \qquad \text{rel } \partial X_{k-1}.$$

We infer that V is a Poincaré cobordism of index k with respect to ∂X_{k-1}. By Theorem E, V is the trace of a sequence of surgeries on a finite collection of Poincaré embeddings $S^{k-1} \times D^{n-k} \subset V$ (here we are using that $3 \leqslant k \leqslant n-3$ and that V is 2-connected). Set $X_k := X_{k-1} \cup V$. Then $X_k \to X$ is a Poincaré embedding representing the k-skeleton of X.

Repeating the above procedure sufficiently many times, we obtain a Poincaré embedding $X_{n-3} \to X$ with $X_{n-3} \simeq K$. It is automatic that X is obtained from X_{n-3} by attaching a single n-cell. This completes the proof.

7. Proof of Theorem C

Assume that $(X^n, \partial X^n)$ is $(2k-n+2)$-connected, X is 2-connected and $k \leqslant n-3$. Given $f: (D^k, S^{k-1}) \to (X, \partial X)$, we wish to show that it is the underlying map of a relative embedded thickening.

If $k \leqslant 2$, we first represent $f_{S^{k-1}}: S^{k-1} \to \partial X$ by an embedded thickening (using, say, the main theorem of [**9**]). Then we can apply Theorem A to extend this to a relative embedded thickening of f. So for the remainder of the proof, assume that $k \geqslant 3$.

When $k \geqslant 3$, the argument is given in two steps. First apply Theorem A to the associated map

$$g: (\partial X \cup_f D^k, \partial X) \to (X, \partial X).$$

This gives a Poincaré duality space thickening W^n of $\partial X \cup_f D^k$ with W codimension zero Poincaré embedded in X such that

$$\partial W = \partial X \amalg \partial_1 W.$$

Moreover, W is an elementary cobordism of index k relative to ∂X.

The next and last step is to apply Theorem D to W. This shows that W is obtained from ∂X by attaching a Poincaré k-handle. Since W is Poincaré embedded in X relative to ∂X, we are done.

Acknowledgements

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