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POINCARÉ DUALITY EMBEDDINGS AND FIBERWISE HOMOTOPY THEORY

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We prove an embedding theorem for maps from a finite complex into a Poincaré duality space. The proof uses fiberwise homotopy theory. © 1999 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

The choice of a closed regular neighborhood of a finite polyhedron embedded in a manifold enables one to write the ambient space as a union of two manifolds, the neighborhood and its complement, glued along a common boundary component. Poincaré duality embeddings are a homotopy theoretic version of this in which the manifolds are replaced by Poincaré spaces and the gluing is now done with respect to a homotopy equivalence.

To put it another way, a map $f: K \to X$ from a finite complex K into an n-dimensional Poincaré duality space X (or Poincaré pair $(X, \partial X)$) is said to *Poincaré embed* if f extends to a homotopy equivalence

$$K \cup_A C \simeq X$$

such that each piece of the decomposition satisfies Poincaré duality. This means that (K, A) and (C, A) (or $(C, \partial X \amalg A)$ when X has a boundary) are Poincaré pairs having the same dimension. Moreover, the fundamental class in each case is induced from the fundamental class of X (for the precise definition, see 2.2 below). Thus to specify the Poincaré embedding, we have to find the *complement* C and the way which it is glued to K to give X.

There are obstructions to Poincaré embedding. For example, for such a decomposition of X to exist it is necessary that the homology of K (with respect to any coefficient system) vanishes in degrees $> n$. Let us write $\operatorname{hodim} K \leqslant k$ if K is homotopy equivalent to a CW complex of dimension $\leqslant k$. We will be working with the codimension $\geqslant 3$ hypothesis: $k \leqslant n - 3$.

Question. Given a map $f: K \to X$, when does it Poincaré embed?

The problem may be broken up into two stages: first construct a candidate for the complement of K, then, provided that the candidate has been correctly chosen, find the gluing data to build X. In this paper we give a partial answer to the above question in terms of a lower bound for the connectivity of f.

Theorem A. *Let $f: K \to X$ be r-connected. Then f Poincaré embeds provided $k \leqslant n - 3$ and*

$$r \geqslant 2k - n + 2\,.$$

This is the Poincaré analogue of a theorem of Wall which gives criteria for a finite complex to embed up to homotopy in a smooth manifold [17]. Actually, Wall's result is one dimension sharper than our Theorem A, and I do not know whether the bound in the Poincaré case can be improved to get the extra dimension. When $r = 0$, Theorem A is a result of Levitt [10]. When $r = 1$, X is 1-connected and K is a Poincaré space, it is a result of Hodgson [6]. The proofs of the theorems of Wall, Levitt and Hodgson are based on engulfing and the Whitney trick.

In contrast, our approach will be homotopy theoretic. The technology developed here should be of interest to both homotopy theorists and manifold topologists. For homotopy theorists, we introduce new tools for studying cubical diagrams of spaces. Unbased fiberwise homotopy theory over a fixed space is discussed and a Freudenthal-type desuspension theorem is proved (which is different from the corresponding *based* version that is to be found in the work of James [7, Section 9]; see 4.10 below). This fiberwise desuspension result enables us to construct the complement.

For manifold theorists, we point out that Theorem A has applications to embedding theory in codimension $\geqslant 3$ via the surgery machine (see e.g. Corollary A below).

We also mention that fiberwise technology is powerful enough so as to classify embeddings in a range which is about twice as large as the one appearing here (the classification is in terms of homotopy theoretic data which are sometimes computable). There is also a companion result to Theorem A which says that the embedding is unique 'up to isotopy' when the bound on connectivity is replaced by strict inequality. The uniqueness result is a special case of a relative version of Theorem A which applies to maps of pairs $(K, L) \to (X, \partial X)$ whose source is a relative CW pair and whose target is a Poincaré pair, where it is already assumed that a Poincaré embedding has been specified for the restriction $L \to \partial X$. The issue of uniqueness, the relative case and classification will be addressed in another paper.

By combining Theorem A with the Browder–Casson–Sullivan–Wall theorem [20, 11.3.2] (and some benign manipulations with Whitehead torsion which we omit) we obtain the following, originally due to Haefliger [5]:

Corollary A. *An r-connected map $f: V^k \to N^n$ of smooth manifolds (with V closed) is homotopic to a smooth embedding provided that $3(k + 1) \leqslant 2n$, $n \geqslant 6$ and $r \geqslant 2k - n + 2$.*

Applying Theorem A to the diagonal map $X \to X \times X$ of a closed Poincaré duality space, we settle in the affirmative an old conjecture in the 2-connected case, which concerns the existence of the unstable homotopy tangent bundle for Poincaré spaces:

Corollary B. *If X is 2-connected, then the diagonal $X \to X \times X$ Poincaré embeds.*

To provide some further context for Theorem A, we now make a few remarks about Poincaré surgery (however, we do not pursue the issue of Poincaré surgery in this article). Suppose that one is given a normal map $g: V^n \to X^n$ of Poincaré spaces, i.e., a degree one map which is covered by a map of Spivak normal fibrations. A basic problem of this subject is to decide when the map is cobordant through normal maps to a homotopy equivalence. A reasonable way to go about this would be to do surgery on framed Poincaré embedded spheres in V to improve the connectivity of g. If say, X is 1-connected and $n \geqslant 5$, then Theorem A allows one to perform a sequence of surgeries on framed Poincaré embedded spheres to obtain an $[n/2]$-connected normal map. Thus Theorem A has application to surgery *below* the middle dimension.

Here is the scheme of the proof of Theorem A. Consider $f: K \to X$ followed by the inclusion $X \subset X \times D^j$. For large j we show that the composite map Poincaré embeds. The next step is to compress this Poincaré embedding down into X. By a downward induction on codimension, it suffices to consider the case $j = 1$. Let W denote the Poincaré complement to K in $X \times I$. The first obstruction to compressing down into X is given by the existence of a map $C \to X$ and a fiberwise weak equivalence $\Sigma_X C \simeq W$, i.e., the complement W needs to fiberwise desuspend over X. It turns out that our bound for the connectivity is sufficient to guarantee that such a fiberwise desuspension exists. The space C will be our candidate for the complement of K in X. Let A' be the space along which K and W are glued to make $X \times I$. The final step of the proof is to show that A' fiberwise desuspends over K in a way compatible with the fiberwise desuspension we chose for W (the resulting desuspension will be our candidate for gluing K to C to build X). Identify A' with $\Sigma_K A$ for some map $A \to K$, and identify W with $\Sigma_X C$. We show there exists a map $A \to C$ such that the resulting composite $\Sigma_K A \to \Sigma_X A \to \Sigma_X C$ coincides with the given map $A' \to W$ via the identifications. Then the resulting data amount to a Poincaré embedding of $f: K \to X$.

Outline. This article is organized as follows: Section 2 sets forth the basic definitions and conventions; most of the material here is well known. In Section 3 we establish the existence of Poincaré embeddings in the stable case: we show that a map $f: K \to X$ followed by the inclusion $X \subset X \times D^j$ will Poincaré embed when j is sufficiently large. In Section 4 we prove the Truncation Lemma 4.1, which is the main technical tool for deducing the Cocartesian Replacement Theorem 4.2 and the Desuspension Theorem 4.7. In Section 5 we prove the Face Theorem 5.1, which is an excision statement about cubical diagrams of spaces. In Section 6 we prove Theorem A.

2. PRELIMINARIES

We will be using the Quillen model category structure on the category **Top** of compactly generated topological spaces [11]. In this model category, the weak equivalences are the weak homotopy equivalences, the fibrations are the Serre fibrations and the cofibrations are the "Serre cofibrations", i.e., inclusion maps given by a sequence of cell attachments (i.e. relative cellular inclusions.) or retracts thereof. In particular every object is fibrant and the cofibrant objects are retracts of cellular objects. Fibrations are specified as "$\twoheadrightarrow$", cofibrations as "$\rightarrowtail$" and weak equivalences as "$\xrightarrow{\sim}$".

Each morphism of **Top** can be functorially factored in two ways as: (1) a cofibration followed by a fibration which is also a weak equivalence, or as (2) a cofibration which is a weak equivalence followed by a fibration. Applying the first of these options to the map $\emptyset \to Y$, we obtain for each object a cofibrant replacement $Y^c \overset{\sim}{\twoheadrightarrow} Y$.

We will be working for the most part with cofibrant objects. Unless otherwise specified, throughout this paper the term "space" will mean a cofibrant object. A space is called *homotopy finite* if it is homotopy equivalent to a finite complex. A map $A \to B$ of spaces (with B nonempty) is *r-connected* if for any choice of basepoint in B, the homotopy fiber with respect to this choice of basepoint is an $(r-1)$-connected space (by convention, a non-empty space is at least (-1)-connected). In particular, any map $A \to B$ is (-1)-connected. A weak equivalence is an ∞-connected map.

If no confusion arises, the following slightly ambiguous notation will often be used: if $f: A \to B$ is a map, we let the pair denoted $(\bar{B}, A)$ consist of the mapping cylinder $\bar{B} = B \cup_f A \times [0,1]$ together with the inclusion of $A \times 0$.

The machinery of homotopy limits and colimits will be used throughout (cf. [2]). We also assume that the reader is familiar with homotopy excision, i.e., the Blakers–Massey theorem and its (dual) generalization to cubical diagrams of spaces. A basic reference for the latter is [4, Section 2].

Lastly, a warning about terminology: suppose we are given a commutative square of spaces

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array}$$

such that the induced map $B \times 0 \cup A \times [0,1] \cup C \times 1 \to D$ is a homotopy equivalence. One usually says in this instance that the square is homotopy cocartesian (or a homotopy pushout). However, we will instead follow Goodwillie's conventions and say that the square is *cocartesian*. Similarly, the square is j-cocartesian if the map is j-connected (thus ∞-cocartesian is the same thing as cocartesian). Analogous terminology will be used in the cartesian case. We also use this terminology for cubical diagrams of spaces.

2.1. Poincaré duality spaces

Let X be a homotopy finite space equipped with a local coefficient system $\mathscr{L}$ (i.e., a functor from the fundamental groupoid of X to the category of abelian groups) which is pointwise free abelian of rank one. Let $[X] \in H_n(X; \mathscr{L})$ be a class. The data $(X, \mathscr{L}, [X])$ equip X with the structure of a *Poincaré duality space of dimension n* if cap product induces an isomorphism

$$\cap [X]: H^*(X; \mathscr{M}) \xrightarrow{\cong} H_{n-*}(X, \mathscr{L} \otimes \mathscr{M})$$

for every local system $\mathscr{M}$. When $\mathscr{L}$ and $[X]$ are understood, we will simply refer to X as a *PD space*.

If π_x denotes the fundamental group at $x \in X$, then the local system which assigns to x the integral group ring $\mathbb{Z}[\pi_x]$ is denoted by Λ. It is a fact that $\cap [X]$ defines an isomorphism for all local systems $\mathscr{M}$ if and only if it does for Λ (cf. [19, 1.1]).

A cofibration pair $(X, \partial X)$ consisting of homotopy finite spaces together with $\mathscr{L}$ and a class $[X] \in H_n(X, \partial X; \mathscr{L})$ will be called a *Poincaré pair of dimension n* if, similarly, cap product induces an isomorphism

$$\cap [X]: H^*(X; \mathscr{M}) \xrightarrow{\cong} H_{n-*}(X, \partial X; \mathscr{L} \otimes \mathscr{M})$$

for all $\mathscr{M}$ and moreover, the restriction of $\mathscr{L}$ to ∂X together with the image of $[X]$ under the boundary homomorphism $H_n(X, \partial X; \mathscr{L}) \to H_{n-1}(\partial X; \mathscr{L})$ equips ∂X with the structure of a PD space. Again, it is enough to check these conditions in the case when $\mathscr{M}$ is Λ. We will refer to Poincaré pairs as *PD pairs*.

There is often redundancy (compare [3, 2.2.3]).

LEMMA 2.1 *If $(X, \partial X)$ is 2-connected and $\cap [X]$ is an isomorphism, then $(X, \partial X)$ is a PD pair.*

Proof. I learned of the following argument from A. Ranicki: let $[\partial X]$ denote the image of $[X]$ with respect to the boundary homomorphism. Since $\pi_1(X) \cong \pi_1(\partial X)$, it will be enough to check that

$$\cap[\partial X]: H^*(\partial X;\Lambda) \to H_{n-*-1}(\partial X;\mathscr{L}\otimes\Lambda)$$

is an isomorphism.

Consider the commutative diagram

$$\begin{tikzcd}
C^*(X,\partial X;\Lambda) \arrow[r] \arrow[d] & C^*(X;\Lambda) \arrow[r] \arrow[d] & C^*(\partial X;\Lambda) \arrow[d] \\
C_{n-*}(X;\mathcal{L}\otimes\Lambda) \arrow[r] & C_{n-*}(X,\partial X;\mathcal{L}\otimes\Lambda) \arrow[r] & C_{n-*-1}(\partial X;\mathcal{L}\otimes\Lambda)
\end{tikzcd}$$

whose horizontal maps induce long exact sequences in homology. The middle and right vertical maps are given by chain level versions of $\cap[X]$ and $\cap[\partial X]$ respectively. The left vertical arrow is also given by a cap product with $[X]$. By hypothesis, the middle vertical map induces a homology isomorphism. Therefore, by the five lemma, it is sufficient to show that the left vertical map induces a homology isomorphism.

The left vertical map can also be obtained as follows: by hypothesis, we have a chain homotopy equivalence

$$\cap[X]: C^{n-*}(X;\mathscr{L}\otimes\Lambda) \xrightarrow{\simeq} C_*(X,\partial X;\Lambda).$$

For a left (right) Λ-module P, let $P^{\#} = \hom_\Lambda(P,\Lambda)$ denote its dual right (left) module given by taking module homomorphisms into Λ. Dualize the map $\cap[X]$ to get another chain homotopy equivalence, $(\cap[X])^{\#}$.

Since $C_*(X,\partial X;\Lambda)$ is (up to homotopy) a chain complex of finitely generated free modules, $C_*(X,\partial X;\Lambda)^{\#}$ is identified with $C^*(X,\partial X;\Lambda)$ (because a finitely generated free module is canonically isomorphic to its double dual). Similarly, $C^{n-*}(X;\mathscr{L}\otimes\Lambda)^{\#}$ is identified with $C_{n-*}(X;\mathscr{L}\otimes\Lambda)$. With respect to these identifications, the map $(\cap[X])^{\#}$ is the left vertical map of the diagram (this follows from the way cap products are constructed). Consequently, the left vertical map of the diagram is a chain homotopy equivalence. □

2.2. *Poincaré duality embeddings*

Definition 2.2. Let $f: K \to X$ denote a map from a connected homotopy finite space K to a PD space X or PD pair $(X,\partial X)$ of dimension n. A *PD embedding* for f is a commutative square of spaces

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & C \arrow[d, "j"] \\
K \arrow[r, "f"'] & X
\end{tikzcd}$$

and a choice of factorization of $\partial X \subset X$ as $\partial X \to C \xrightarrow{j} X$, such that:

- The square is cocartesian.
- The spaces A and C are homotopy finite.

- The image of the fundamental class $[X]$ under the composite

$$H_n(X,\partial X)\cong H_n(\bar{X},\partial X)\to H_n(\bar{X},C)\cong H_n(\bar{K},A)$$

  equips $(\bar{K},A)$ with the structure of a PD pair (here we are suppressing the local systems in the notation). Similarly, the image of $[X]$ with respect to the evident map $H_n(X,\partial X)\to H_n(\bar{C},\partial X \amalg A)$ equips $(\bar{C},\partial X \amalg A)$ with the structure of a Poincaré pair.
- If hodim $K\leqslant k$, then $A\to K$ is $(n-k-1)$-connected.

The space C is called the *complement*, and A is called the *gluing space*. If there exists a PD embedding for f, then we say that *f PD embeds*. If hodim $K\leqslant k$, then we say that the *codimension* of the embedding is $\geqslant n-k$.

Again, there is often some redundancy:

LEMMA 2.3. *If* hodim $K\leqslant n-3$, *and all of the conditions of the definition are known to hold except perhaps duality for the pair* $(\bar{C},\partial X\amalg A)$, *then the diagram is a PD embedding.*

Proof. This is essentially proved in [20, 2.7ii,11.1], but with a missing hypothesis. We need to establish that $(\bar{C},\partial X \amalg A)$ is a Poincaré pair. For this, it will be enough to check that

$$\cap[C]\colon H^*(C)\to H_{n-*}(\bar{C},\partial X \amalg A)$$

is an isomorphism, where $[C]\in H_n(\bar{C},\partial X \amalg A)$ is obtained from $[X]$ as indicated in the definition. By the cohomology exact sequence

$$\cdots\to H^*(\bar{C},\partial X \amalg A)\to H^*(\bar{C},A)\to H^*(\partial X)\to\cdots$$

and the naturality of cap product, it suffices to show that

$$\cap[C]\colon H^*(\bar{C},A)\to H_{n-*}(\bar{C},\partial X)$$

is an isomorphism (since ∂X is a PD space).

The exact sequence

$$\cdots\to H^*(\bar{C},A)\to H^*(X)\to H^*(K)\to\cdots$$

associated with the cocartesian square, the naturality of cap product, and the fact that $(X,\partial X)$ and $(\bar{K},A)$ are PD pairs implies that $\cap[C]$ is an isomorphism. □

Remarks 2.4 (1). Assume that K is a PD space of dimension k with $k\leqslant n-3$. Then the map $A\to K$ has an $(n-k-1)$-spherical homotopy fiber over any point (by [15, 4.4], [3, I.4.3]). In this situation $A\to K$ plays the role of normal bundle for K in X. This particular kind of PD embedding is discussed in [20, Chap. 11].

(2). PD embeddings arise from manifold embeddings in the following way. Suppose that V is a closed regular neighborhood of a k-dimensional finite connected polyhedron embedded in the interior of a compact n-manifold N. Let C be the closure of $N\setminus V$, and let A be the boundary of V. Then $N=V\cup_A C$, and the data determine a PD embedding of V in N.

(3). If k is an integer such that hodim $K\leqslant k\leqslant n-3$, then to check that $A\to K$ is $(n-k-1)$-connected, it is sufficient to know that it is 2-connected, once we know that $(\bar{K},A)$ satisfies n-dimensional Poincaré duality. The reason this is true is that duality implies the homology of $(\bar{K},A)$ (with respect to any coefficient system) will vanish in degrees

$\leqslant n-k-1$. The relative Hurewicz theorem [21, 7.2] then shows that the relative homotopy groups will also vanish in this range when $A \to K$ is 2-connected.

In any case, the assumption that $A \to K$ is $(n-k-1)$-connected arises from geometry: if the PD embedding arises from a manifold embedding as in (2) above, then this connectivity is a consequence of transversality.

The next lemma concerns the extent to which the notion of PD embedding is homotopy invariant.

LEMMA 2.5. *Suppose that* $f: K \to X$ *PD embeds. Then*

(1). *If* g *is homotopic to* f, *then* g *PD embeds.*

(2). *Let* $\rho: L \xrightarrow{\sim} K$ *be a homotopy equivalence. Then* $f \circ \rho: L \to X$ *PD embeds.*

(3). *Let* $h: (X, \partial X) \xrightarrow{\sim} (Y, \partial Y)$ *be a homotopy equivalence. Then* $h \circ f: K \to Y$ *PD embeds.*

Proof. Let

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[d] \\ K \arrow[r, "f"'] & X \end{tikzcd}$$

be a PD embedding.

(1). Replace f by g and C by the mapping cylinder $\bar{C}$ of $A \amalg \partial X \to C$. A choice of homotopy from f to g induces a map $\bar{C} \to X$ which defines the desired PD embedding for g.

(2). Let $\rho^{-1}: K \to L$ be a choice of homotopy inverse for ρ. Then the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\rho^{-1} \circ i"'] & C \arrow[d] \\ L \arrow[r, "f \circ \rho"'] & X \end{tikzcd}$$

is homotopy commutative. As in the first part, replace C by a suitable mapping cylinder to get the desired PD embedding of $f \circ \rho$.

(3). By taking an appropriate mapping cylinder, we can assume that the map $A \amalg \partial X \to C$ is a cofibration. Let C' denote the space

$$(A \amalg \partial Y) \cup_{A \amalg \partial X} C \ .$$

Then there is an evident PD embedding

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C' \arrow[d] \\ K \arrow[r, "h \circ f"'] & Y \end{tikzcd} .$$

2.3. Stabilization

Let $S(\xi) \to X$ denote a $(j-1)$-spherical fibration with $S(\xi)$ not necessarily cofibrant. Even if $S(\xi)$ were cofibrant, the restriction $S(\xi|Z)$ of $S(\xi)$ along a cofibration $Z \rightarrowtail X$ need not be. For this reason, we introduce the following technical innovation: let $\flat$: **Top** $\to$ **Top** be the functor which maps a space to the geometric realization of its total singular complex. Then $\flat$ is pointwise equivalent to the identity. Furthermore, $\flat$ applied to a monomorphism gives a cofibration. If $F: J \to \mathbf{Top}$ denotes a finite diagram, let $\mathrm{hocolim}^\flat F$ denote the effect of first applying $\flat$ pointwise and then taking the resulting homotopy colimit.

With respect to this convention, let $\partial D(\xi)$ be defined as the $\mathrm{hocolim}^\flat$ of the diagram

$$\partial X \leftarrow S(\xi|\partial X) \to S(\xi)\,.$$

Similarly, Let $D(\xi)$ be defined as $\mathrm{hocolim}^\flat$ of the diagram

$$X \leftarrow S(\xi) \xrightarrow{=} S(\xi)$$

(equivalently, the mapping cylinder of the map $\flat(S(\xi)) \to \flat(X)$).

Then $(D(\xi), \partial D(\xi))$ is a PD pair of dimension $n + j$ (the orientation and fundamental class are induced from the ones on $(X, \partial X)$ via the Thom isomorphism). This construction is the Poincaré analogue of replacing an n-manifold with boundary by the total space of a j-disk bundle which lies over it.

Given a PD embedding

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ K \arrow[r, "f"'] & X \end{tikzcd},$$

we shall construct another PD embedding whose target is $D(\xi)$. Let $\Sigma^\xi C$ be the space

$$\mathrm{hocolim}^\flat(C \leftarrow S(\xi|C) \to S(\xi))$$

and similarly, let $\Sigma^{\xi|K} A$ be the space

$$\mathrm{hocolim}^\flat(A \leftarrow S(\xi|A) \to S(\xi|K))\,.$$

Then these assemble to a PD embedding

$$\begin{tikzcd} \Sigma^{\xi|K} A \arrow[r] \arrow[d] & \Sigma^\xi C \arrow[d] \\ D(\xi|K) \arrow[r] & D(\xi) \end{tikzcd}$$

where $D(\xi|K)$ means $\mathrm{hocolim}^\flat$ of the diagram $K \leftarrow S(\xi|K) \to S(\xi|K)$. In particular, K and $D(\xi|K)$ are canonically homotopy equivalent, so by Lemma 2.5(2), we obtain a PD embedding

$$\begin{tikzcd} \Sigma^{\xi|K} A \arrow[r] \arrow[d] & \Sigma^\xi C \arrow[d] \\ K \arrow[r] & D(\xi) \end{tikzcd}.$$

A special case of this construction occurs when $S(\xi) \to X$ is the trivial fibration with fiber S^0. If this is the case, the pair $(D(\xi), \partial D(\xi))$ identifies with $(X \times I, \partial(X \times I))$, and the new PD embedding is called the *decompression*. It increases the codimension by one, and is the Poincaré analogue of the standard way of passing from an embedding in a manifold to one in the product of the manifold with the unit interval.

In this instance, $\Sigma^{\xi}C$ is a variant of the *fiberwise suspension* of $C \to X$. This fiberwise suspension, denoted $\Sigma_X C$, is given by the double mapping cylinder

$$X \times 0 \cup C \times [0,1] \cup X \times 1 .$$

the map $\partial(X \times I) \to X \times I$ factors canonically through $\Sigma_X C$. Note that $\partial(X \times I)$ is just $\Sigma_X \partial X$.

With respect to this variant of the construction, the PD embedding becomes

$$\begin{array}{ccc} \Sigma_K A & \longrightarrow & \Sigma_X C \\ \big\downarrow & & \big\downarrow \\ K & \longrightarrow & X \times I \end{array} .$$

Notes 2.6. The basic reference for much of the material in this section is Wall's foundational paper [19]. For a recent survey about Poincaré duality spaces, see [9]. The definition of PD embedding given here is very similar to the one proposed by Levitt [10, 2.3].

3. EXISTENCE OF STABLE PD EMBEDDINGS

Given a PD pair $(X, \partial X)$ of dimension n, the cartesian product with a disk D^j yields a PD pair $(X \times D^j, \partial(X \times D^j))$ of dimension $n + j$, where $\partial(X \times D^j)$ is the amalgamated union $X \times S^{j-1} \cup (\partial X) \times D^j$.

Given a map $f: K \to X$, we let f also denote the composition

$$K \xrightarrow{f} X \subset X \times D^j$$

where the second of these maps is given by identifying X with $X \times 0$ by means of the identity.

LEMMA 3.1. *There exists a positive integer j such that $f: K \to X \times D^j$ PD embeds.*

Proof. Suppose first that $(X, \partial X)$ has the homotopy type of a PL manifold with boundary. If this holds, we may further assume that $(X, \partial X)$ is actually a PL manifold, by 2.5(3). Take the cartesian product with a suitably large disk D^j, and use general position to replace $f: K \to X \times D^j$ by an embedding up to homotopy (see e.g. [17]). Thus there is a codimension zero compact submanifold N in the interior of $X \times D^j$ and a homotopy equivalence $K \simeq N$ such that the composite $K \simeq N \subset X \times D^j$ coincides with f up to homotopy. Applying 2.5(1), obtain the desired PD embedding.

Now assume that $(X, \partial X)$ is general. By regular neighborhood theory [14, Chap. 3], for some $t \gg n$ there exists a compact PL manifold $N^{n+t} \subset \mathbb{R}^{n+t}$ equipped with a decomposition of its boundary $\partial N = \partial_- N \cup_{\partial_0 N} \partial_+ N$, and a homotopy equivalence of pairs $(X, \partial X) \simeq (N, \partial_- N)$. Then the homotopy fiber of the map $\partial_+ N \to N$ is an $(n+t-1)$-sphere (see [15, 4.4], [3, I.4.1]).

By the previous case, we can assume that the composite

$$K \to X \xrightarrow{\sim} N$$

PD embeds.

Choose a fiber homotopy inverse $S(\xi) \to N$ for $\partial_+ N \to N$ in the reduced Grothendieck group of spherical fibrations over N. Suppose that the fiber of $S(\xi) \to N$ is $S^{\ell-1}$.

Stabilizing with respect to $S(\xi) \to X$, we obtain a PD embedding

$$\begin{tikzcd} \Sigma^{\xi|K} U \arrow[r] \arrow[d] & \Sigma^{\xi} C \arrow[d] \\ K \arrow[r] & D(\xi) \end{tikzcd} .$$

It is straightforward to check that there is a homotopy equivalence of pairs

$$(D(\xi), \partial D(\xi)) \simeq (X \times D^{t+\ell}, \partial(X \times D^{t+\ell}))$$

in such a way that the map $K \to D(\xi)$ corresponds to f up to homotopy. Applying 2.5(1) completes the proof. $\square$

Notes 3.2. This is the only argument of the paper which uses manifolds. However, there is an alternative proof which is entirely homotopy theoretic. The alternative argument requires the technology of fiberwise/equivariant duality. For reasons of space we relegate this to another paper.

We have implicitly used the Spivak normal fibration of $(X, \partial X)$ in the proof of Lemma 3.1. For a homotopy theoretic proof of the existence of the Spivak fibration, see [8], [9].

4. TRUNCATION, COCARTESIAN REPLACEMENT, AND FIBERWISE DESUSPENSION

4.1. The truncation lemma

Let π be a group. Let P be a based connected space with fundamental group π. Let $U \to P$ be a map of spaces. Let K_* be a chain complex of projective $\mathbb{Z}[\pi]$-modules. Lastly, let $C_*(P, U) \to K_*$ be a $\mathbb{Z}[\pi]$-linear chain map, where $C_*(P, U)$ is the free chain complex of $\mathbb{Z}[\pi]$-modules which computes the relative homology of $U \to P$.

Assume that $\dim K_* \leqslant n$ in the sense that its cohomology (for any coefficient module) vanishes in degrees $> n$. Assume that the chain map $C_*(P, U) \to K_*$ is n-connected.

LEMMA 4.1 (Truncation). *If $n \geqslant 2$ there exists a factorization*

$$U \to A \to P$$

such that

- *$C_*(A, U) \to K_*$ is a chain homotopy equivalence.*
- *$U \to A$ is a relative CW complex of* dim $\leqslant n$.
- *$A \to P$ is $(n-1)$-connected.*

Proof. For this proof, chain complexes and homology are understood to be taken with respect to the coefficient module $\mathbb{Z}[\pi]$.

Factor $U \to P$ as $U \rightarrowtail W \to P$ where (W, U) is a relative CW complex of dimension $\leqslant n-1$ and the map $W \to P$ is $(n-1)$-connected. The chain map $C_*(W, U) \to K_*$ is $(n-1)$-connected. The cohomology of its mapping cone with respect to any coefficient module vanishes in degrees $> n$, (since $\dim K_* \leqslant n$ and $\dim C_*(W, U) \leqslant n - 1$). The homology of the mapping cone vanishes in degrees $< n$. Therefore, by an observation of Wall, homology of the mapping cone in degree n is a projective $\mathbb{Z}[\pi]$-module (see [28, 2.3] or the proof of [16, 2.1]). Call it Q.

Case (1): *Q is free.* Choose a basis for Q. The long exact sequence gives a surjection $H_n(\bar{P}, W) \to Q$. The relative Hurewicz theorem gives a surjection $\pi_n(\bar{P}, W) \to H_n(\bar{P}, W)$. Choose a lift for each basis element of Q and attach n-cells to W corresponding to these lifts. Call the resulting space A. Then A gives the desired factorization.

Case (2): *Q is arbitrary.* At the cost of adding cells we can make Q free as follows: let Q' be such that $Q \oplus Q'$ is free. Let F be the free module $Q' \oplus Q \oplus Q' \oplus \cdots$. Then $Q \oplus F \cong F$, by the Eilenberg Swindle. Attach $(n-1)$-cells to W in a trivial way indexed by a basis for F. Let W' be the result of this procedure. Extend $W \to P$ to W' by mapping the new cells to the basepoint of P. The relative of homology of $W' \to P$ is again concentrated in degree in n and it is isomorphic to $Q \oplus F$. Case (1) now applies. $\square$

We next give some applications.

4.2. Cocartesian replacement

Let

$$\begin{array}{ccc} X_\emptyset & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_{12} \end{array}$$

be a commutative square of connected based spaces. We shall provide criteria for deciding when it is possible to replace $X_\emptyset$ by another space such that the new square is cocartesian.

Let $\pi = \pi_1(X_{12})$. Assume that

(1). The square is j-cocartesian for some $j \geqslant 3$.
(2). The homomorphism $\pi_1(X_\emptyset) \to \pi$ is an isomorphism.
(3). The relative cohomology (with any $\mathbb{Z}[\pi]$-module coefficients) of the map $X_1 \vee X_2 \to X_{12}$ vanishes in degrees $> j$.

THEOREM 4.2. *Under the above assumptions, there exists a based space A and a based map $A \to X_\emptyset$ such that the resulting diagram*

$$\begin{array}{ccc} A & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_{12} \end{array}$$

is cocartesian. Furthermore, the map $A \to X_\emptyset$ can be chosen as $(j-2)$-connected.

Proof. Let K_* denote the $\mathbb{Z}[\pi]$-module chain complex given by

$$\operatorname{holim}(C_*(X_1) \to C_*(X_{12}) \leftarrow C_*(X_2))$$

(equivalently, the desuspension of the mapping cone of $C_*(X_1) \oplus C_*(X_2) \to C_*(X_{12})$). Then the map

$$C_*(X_\emptyset) \to K_*$$

is $(j-1)$-connected. Furthermore, the cohomology of K_* with respect to any coefficient system vanishes in degrees $> j-1$. Applying the Truncation Lemma 4.1 (with $U = \emptyset$), we obtain a map $A \to X_\emptyset$ with the desired properties. (To check that the resulting square is cocartesian, use the Whitehead theorem in conjunction with the fact that the map $\operatorname{hocolim}(X_1 \leftarrow A \to X_2) \to X_{12}$ induces an isomorphism of fundamental groups and an isomorphism in cohomology with respect to any coefficient module.) □

We also have a relative version:

Addendum 4.3. *With a square satisfying conditions* (1) *and* (2) *above* (*cf. before Theorem* 4.2), *let* $Z \to X_\emptyset$ *be a map of spaces* (*where* Z *is not necessarily based*). *Suppose instead of* (3) *that the map*

$$\operatorname{hocolim}(X_1 \leftarrow Z \to X_2) \to X_{12}$$

has vanishing relative cohomology in degrees $> j$ (*with respect to any* $\mathbb{Z}[\pi]$*-module coefficients*).

Then there exists a space A *and a factorization* $Z \to A \to X_\emptyset$ *such that the square given by replacing* $X_\emptyset$ *with* A *is cocartesian. Furthermore* $A \to X_\emptyset$ *can be chosen as* $(j-2)$*-connected.*

Clearly, this specializes to Theorem 4.2 by taking Z to be a point.

Proof. Let K_* be the chain complex given by taking the mapping cone of the map

$$C_*(Z) \to \operatorname{holim}(C_*(X_1) \to C_*(X_{12}) \leftarrow C_*(X_2))\,.$$

Apply Lemma 4.1 to the evident map

$$C_*(\bar{X}_\emptyset, Z) \to K_*\,.$$

This gives a factorization $Z \to A \to X_\emptyset$ such that the composite

$$C_*(\bar{A}, Z) \to C_*(\bar{X}_\emptyset, Z) \to K_*$$

is a chain equivalence. For this choice of A, the new square is cocartesian.

4.3. Fiberwise desuspension

Let $f: A \to X$ be a map of spaces. Let $\mathbf{Top}_f$ be the category in which an *object* is specified by a factorization $A \to Y \to X$. A *morphism* is a map $Y \to Z$ which preserves factorizations. A morphism is a *weak equivalence*, *fibration* or *cofibration* if it respectively is so when considered in $\mathbf{Top}$ by means of the forgetful functor $\mathbf{Top}_f \to \mathbf{Top}$.

Lemma 4.4. *With respect to the above conventions*, $\mathbf{Top}_f$ *is a model category.*

Proof. For any model category C, Quillen [11, II.2.8] shows that the over category $C_{/X}$ and the opposite category C^{op} are also model categories. The weak equivalences and fibrations of $C_{/X}$ are defined via the forgetful functor $C_{/X} \to C$. The weak equivalences and the cofibrations of C^{op} correspond to the weak equivalences and fibrations of C.

The map $f: A \to X$ defines an object of $\mathbf{Top}_{/X}$. Denote it by $[f]$. Then $\mathbf{Top}_f$ is isomorphic to

$$((\mathbf{Top}_{/X})^{\mathrm{op}}_{/[f]^{\mathrm{op}}})^{\mathrm{op}}$$

and the result follows by remarks in the previous paragraph. □

Remark 4.5. An object $Y \in \mathbf{Top}_f$ is fibrant when the structure map $Y \to X$ is a Serre fibration. It is cofibrant when the structure map $A \to Y$ is a Serre cofibration.

We will assume in what follows that X is a connected space. Using the above conventions, we shall regard *fiberwise suspension* as a functor

$$\Sigma_X: \mathbf{Top}_{/X} \to \mathbf{Top}_{\nabla},$$

where $\nabla: X \amalg X \to X$ is the fold map. It is straightforward to check that Σ_X maps cofibrant objects to cofibrant objects.

Definition 4.6. An object $Y \in \mathbf{Top}_{\nabla}$ is *j-connected* if the structure map $Y \to X$ is a $(j+1)$-connected map of topological spaces. We will say that Y has *dimension* $\leqslant n$ if the structure map $X \amalg X \to Y$ has the property that its relative cohomology (with respect to the pullback of any local system on X along $Y \to X$) vanishes in degrees $> n$.

THEOREM 4.7 (Desuspension). *Let $Y \in \mathbf{Top}_{\nabla}$ be a fibrant and cofibrant object which is j-connected and has dimension $\leqslant 2j+1$, for some integer $j \geqslant 1$. Then there exists an object $A \in \mathbf{Top}_{/X}$ and a weak equivalence*

$$\Sigma_X A \xrightarrow{\sim} Y.$$

Moreover, the map $A \to X$ can be chosen as j-connected.

Proof. Let $i_\pm : X \to Y$ be the maps obtained by restricting the structure map $X \amalg X \to Y$ to each summand. Let X_- be the effect of factorizing the map i_- as $X \overset{\sim}{\rightarrowtail} X_- \twoheadrightarrow Y$ and let X_+ be defined similarly using i_+. We have a cartesian square

$$\begin{array}{ccc} B & \longrightarrow & X_+ \\ \downarrow & & \downarrow \\ X_- & \longrightarrow & Y \end{array},$$

where $B := X_- \times_Y X_+$ denotes the fiber product of i_- and i_+. Each map in this square is at least 2-connected. Furthermore, the square is $(2j+1)$-cocartesian by the dual Blakers–Massey excision theorem [4, p. 309].

The map $X_- \amalg X_+ \to Y$ has vanishing relative cohomology in degrees $> 2j+1$, so we may apply Theorem 4.2 to conclude that there exists a $(2j-1)$-connected map of spaces $A \to B$ such that the square

$$\begin{array}{ccc} A & \longrightarrow & X_+ \\ \downarrow & & \downarrow \\ X_- & \longrightarrow & Y \end{array}$$

is cocartesian. Make A an object of $\mathbf{Top}_{/X}$ by means of the composite $A \to Y \to X$.

There is an evident chain of weak equivalences of $\mathbf{Top}_\nabla$ given by

$$\Sigma_X A = X\times 0 \cup A\times[0,1]\cup X\times 1 \xleftarrow{\sim} X_-\times 0\cup A\times[0,1]\cup X_+\times 1 \xrightarrow{\sim} Y\,.$$

The proof is completed using a well-known general fact about model categories: an isomorphism in the homotopy category from a cofibrant object to a fibrant object always lifts to a weak equivalence. □

Remarks 4.8. Theorem 4.7 (and its relative version 4.9 below) will be used to construct the complement for the Poincaré embedding in the proof of Theorem A. Richter has pointed out that one can get by with slightly less. Namely, the above cocartesian square involving A, X_-, X_+ and Y can be inserted into the proof of Theorem A instead of the choice of fiberwise desuspension. This replacement would be one way of removing the fiberwise homotopy theory in this paper, but for aesthetic reasons we refrain from doing so.

Here is the relative version of Theorem 4.7.

Addendum 4.9. *Let $\Sigma_X Z \rightarrowtail Y$ be a cofibration of $\mathbf{Top}_\nabla$ for some cofibrant object $Z\in\mathbf{Top}_{/X}$. Assume that*

- *the relative cohomology of the underlying map vanishes in degrees $> 2j+1$ for $j\geqslant 1$ (for all coefficient systems).*
- *The object Y is j-connected.*

Then there exists a cofibrant object $A\in\mathbf{Top}_{/X}$, a morphism $Z\to A$, and a weak equivalence

$$\Sigma_X A \xrightarrow{\sim} Y$$

which is relative to $\Sigma_X Z$. Moreover, the map $A\to X$ can be chosen as j-connected.

Proof. Follow the proof of Theorem 4.7, but work relative to Z and use Addendum 4.3. □

Notes 4.10. Special cases of the Truncation Lemma 4.1 are to be found in the literature. The first result in this direction that I know of is in a paper by Berstein and Hilton [1, Theorem 2.1], who in effect prove a version of Lemma 4.1 when π is trivial. Richter [12] had a version of Theorem 4.2 when X is simply connected.

The Desuspension Theorem 4.7 reduces to the usual Freudenthal suspension theorem when X is a point. On the other hand, Theorem 4.7 is not the kind of fiberwise desuspension result that has appeared in the fiberwise topology literature. The latter falls under the rubric of *based* fiberwise homotopy theory, and concerns the extent to which the *reduced suspension* functor

$$\Sigma_X:\mathbf{Top}_{\mathrm{id}_X}\to\mathbf{Top}_{\mathrm{id}_X}$$

is surjective on the level of homotopy categories.

Incidentally, our relative version Addendum 4.9 contains both the based and unbased variants as extreme cases, where we desuspend relative to either the initial or terminal object of $\mathbf{Top}_{/X}$. Taking Z to be the empty space, we obtain Theorem 4.7. When $Z = X$, we obtain the based result.

Although there are two different forgetful functors $\mathbf{Top}_\nabla\to\mathbf{Top}_{\mathrm{id}_X}$, the based and unbased suspensions are generally very different. For example, take $X = S^1$. Consider the non-trivial bundle $\Sigma\to S^1$ with fiber S^1, where Σ is the Klein bottle. Then $\Sigma = \Sigma_{S^1}S^1$, where we fiberwise suspend the multiplication by 2 map $S^1\to S^1$. But the multiplication by 2 map does not admit a section. This example shows that there are objects of $\mathbf{Top}_V$ which fiberwise desuspend in the unbased sense but which fail to do so in the based sense.

5. THE FACE THEOREM

We now prove a technical result which concerns the degree to which the faces in a cartesian 3-cube are cocartesian. The result will be crucial in the proof of Theorem A.

Let

$$\begin{tikzcd}
& X_\emptyset \arrow[rr] \arrow[dl] \arrow[dd] & & X_3 \arrow[dl] \arrow[dd] \\
X_1 \arrow[rr, crossing over] \arrow[dd] & & X_{13} & \\
& X_2 \arrow[rr] \arrow[dl] & & X_{23} \arrow[dl] \\
X_{12} \arrow[rr] & & X_{123} \arrow[from=uu, crossing over] &
\end{tikzcd}$$

be a commutative 3-cube of spaces.

THEOREM 5.1. *Suppose that the* 3*-cube is cartesian and that*

- *the spaces X_S are connected for each non-empty $S \subset \{1,2,3\}$;*
- *each two dimensional face which meets X_{123} is cocartesian;*
- *the maps $X_j \to X_{ij}$ are k_i-connected and the maps $X_i \to X_{ij}$ are k_j-connected, for $1 \leqslant i < j \leqslant 3$.*

Then each of the squares

$$\begin{tikzcd}
X_\emptyset \arrow[r] \arrow[d] & X_j \arrow[d] \\
X_i \arrow[r] & X_{ij}
\end{tikzcd}$$

is $(k_1+k_2+k_3)$-cocartesian for $1 \leqslant i < j \leqslant 3$. Furthermore, if $k_1 + k_2 + k_3 \geqslant 1$, then $X_\emptyset$ is non-empty. If two of the integers k_i are $\geqslant 1$, then $X_\emptyset$ is connected.

Remark 5.2. Here is the why the result is true on the level of ordinary homology. Call the 3-cube $X_\bullet$ and rewrite it as a map of squares $Y_\bullet \to Z_\bullet$. Let $H_*(X_\bullet)$ mean the reduced homology of the iterated homotopy cofiber of $X_\bullet$. This measures the extent to which $X_\bullet$ fails to be cocartesian on the level of homology.

For general reasons, there is a long exact sequence

$$\cdots \to H_*(Y_\bullet) \to H_*(Z_\bullet) \to H_*(X_\bullet) \to \cdots .$$

By hypothesis, $H_*(Z_\bullet)$ is trivial. The dual Blakers – Massey theorem for 3-cubes [4, Theorem 2.6] implies that $H_*(X_\bullet)$ vanishes in degrees $\leqslant k_1 + k_2 + k_3 + 1$. This shows that $H_*(Y_\bullet)$

vanishes in degrees $\leqslant k_1 + k_2 + k_3$, which is what the theorem asserts on the level of homology.

Proof of Theorem 5.1. We can map the 3-cube $X_\bullet$ to another 3-cube by a pointwise weak equivalence such that every map in the new cube is a fibration. So without loss in generality, we will assume that the maps of $X_\bullet$ are all fibrations.

If $k_1 + k_2 + k_3 \geqslant 1$, then Remark 5.2. shows that $H_*(Y_\bullet)$ vanishes in degrees $\leqslant 1$. A straightforward argument involving the Mayer–Vietoris sequence implies that $X_\emptyset$ is non-empty.

If two of the k_i, say $k_1, k_2 \geqslant 1$, then the map $H_*(X_\emptyset) \to H_*(X_1)$ is an isomorphism in degree 0, because $X_2 \to X_{12}$ is 1-connected and the square containing $X_\emptyset$, X_1, X_2 and X_{12} is homologically 2-cocartesian (again by Remark 5.2). It follows that $X_\emptyset$ is connected (since X_1 is connected).

We now prove the part of the statement concerning the degree to which the 2-faces meeting $X_\emptyset$ are cocartesian. Choose any 2-face of $X_\bullet$ which meets $X_\emptyset$, say

$$\begin{tikzcd} X_\emptyset \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X_{12} \end{tikzcd}.$$

It will be enough to show that this square is $(k_1+k_2+k_3)$-cocartesian. Without loss in generality, we can assume that $k_i \geqslant 0$. Call this square $Y_\bullet$. We consider two cases.

Case (1). $k_3 = 0$.

In this instance we are asking whether $Y_\bullet$ is (k_1+k_2)-cocartesian. The 2-face opposite to $Y_\bullet$ in the 3-cube $X_\bullet$ is cocartesian (the one involving X_2, X_3, X_{23} and X_{123}), so by the Blakers–Massey theorem for squares [4, p. 309] the latter 2-face is (k_1+k_2-1)-cartesian.

But this implies that $Y_\bullet$ is also (k_1+k_2-1)-cartesian, since $X_\bullet$ is cartesian (compare [4, Proposition 1.6]). Applying the dual Blakers–Massey theorem, [4, Theorem 2.6] we infer that $Y_\bullet$ is (k_1+k_2)-cocartesian. This completes case (1).

Case (2). $k_3 \geqslant 1$.

The square $Y_\bullet$ factorizes into four squares:

$$\begin{tikzcd} X_\emptyset \arrow[r] \arrow[d] & X_1 \times_{X_{13}} X_3 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \times_{X_{23}} X_3 \arrow[r] \arrow[d] & X_{12} \times_{X_{123}} X_3 \arrow[r] \arrow[d] & X_{12} \times_{X_{123}} X_{13} \arrow[d] \\ X_2 \arrow[r] & X_{12} \times_{X_{123}} X_{23} \arrow[r] & X_{12} \end{tikzcd}$$

where the new spaces introduced are all fiber products. Let us give each of the newer squares a name: the square on the upper left will be denoted (I), the one on the upper right (II), lower left (III) and lower right (IV). It will be enough to show that each of the squares (I)–(IV) is $(k_1+k_2+k_3)$-cocartesian.

CLAIM 5.3. *The square* (*IV*) *is* ∞-*cocartesian.*

The square (IV) is obtained from the ∞-cocartesian square

$$\begin{tikzcd} X_3 \arrow[r] \arrow[d] & X_{13} \arrow[d] \\ X_{23} \arrow[r] & X_{123} \end{tikzcd}$$

by taking the pullback of its spaces along the map $X_{12} \to X_{123}$. This procedure preserves the degree to which a square is cocartesian, so Claim 5.3 follows.

CLAIM 5.4. *The squares* (*II*) *and* (*III*) *are* $(k_1+k_2+k_3)$-*cocartesian.*

First note a general fact: let $A_\bullet \to C_\bullet$ and $B_\bullet \to C_\bullet$ be morphisms of squares of spaces such that each of the squares $A_\bullet$, $B_\bullet$ and $C_\bullet$ is cartesian and a pointwise fibration (i.e., $A_S \to C_S$ is a weak equivalence and fibration for all $S \subset \{0,1\}$). Then the square $A_\bullet \times_{C_\bullet} B_\bullet$ (given by $A_S \times_{C_S} B_S$) is also cartesian.

A straightforward check (which we omit) shows that the square (II) is obtained in this fashion. Consequently, (II) is cartesian. In particular, the map $X_1 \times_{X_{13}} X_3 \to X_1$ is k_1-connected (since it opposes the map $X_3 \to X_{13}$ in (II)). Similarly, the map $X_1 \times_{X_{13}} X_3 \to X_{12} \times_{X_{123}} X_3$ is (k_2+k_3-1)-connected, since its connectivity may be identified that of $X_1 \to X_{12} \times_{X_{123}} X_{13}$, and the latter map is (k_2+k_3-1)-connected, by the Blakers–Massey theorem applied to the square involving X_1, X_{12}, X_{13} and X_{123}. Claim 5.4 for (II) now follows by applying the dual Blakers–Massey theorem. The argument for the square (III) is similar, and will therefore be omitted.

CLAIM 5.5. *The square* (*I*) *is* $(k_1+k_2+k_3)$-*cocartesian.*

The square (I) is cartesian, since $X_\bullet$ is (the homotopy limit of $X_\bullet$ with $X_\emptyset$ deleted coincides with the homotopy limit of (I) with $X_\emptyset$ deleted). As the map $X_1 \times_{X_{13}} X_3 \to X_{12} \times_{X_{123}} X_3$ is (k_2+k_3-1)-connected (see Claim 5.4 above), we infer (using the cartesian-ness of (I)) that the map $X_\emptyset \to X_2 \times_{X_{23}} X_3$ is also (k_2+k_3-1)-connected.

The map $X_\emptyset \to X_1 \times_{X_{13}} X_3$ is (k_1+k_3-1)-connected (this can be seen as follows: The squares (I) and (III) taken together are cartesian, and the parallel map $X_2 \to X_{12} \times_{X_{123}} X_{23}$ is (k_1+k_3-1)-connected, by the Blakers–Massey theorem for the cocartesian square involving X_2, X_{12}, X_{23} and X_{123}).

It follows by the dual Blakers–Massey theorem that the square (I) is

$$(k_2 + k_3 - 1) + (k_1 + k_3 - 1) + 1 = k_1 + k_2 + 2k_3 - 1$$

cocartesian. By assumption $k_3 \geqslant 1$, so the displayed integer is at least $k_1 + k_2 + k_3$. This establishes Claim 5.5, and completes the proof of Proposition 5.1. $\square$

The following lemma will be used in the next section.

LEMMA 5.6. *Let*

$$\begin{tikzcd} X_\emptyset \arrow[r] \arrow[d] & X_2 \arrow[d] \\ X_1 \arrow[r] & X_{12} \end{tikzcd}$$

be a commutative square of spaces.

(1). *If the diagram is cocartesian and the map $X_\emptyset \to X_2$ is r-connected, then the map $X_1 \to X_{12}$ is also r-connected.*

(2). *Assume that the diagram is cocartesian. If the map $X_\emptyset \to X_1$ is 2-connected and the map $X_1 \to X_{12}$ is s-connected, then the map $X_\emptyset \to X_2$ is also s-connected.*

Proof. (1). By homotopy invariance, we can assume that X_2 is obtained from $X_\emptyset$ by attaching cells of dimension $> r$. Then up to homotopy, X_{12} obtained from X_1 by attaching cells of dimension $> r$. Hence $X_1 \to X_{12}$ is r-connected.

(2). The assertion is trivial if $s \leqslant -1$. We now argue by induction. Suppose that the result holds for some $s \geqslant -1$, and let $X_1 \to X_{12}$ be $(s+1)$-connected. It follows by the induction hypothesis that the map $X_\emptyset \to X_2$ is s-connected. Let r be the connectivity of the map $X_\emptyset \to X_1$. The Blakers–Massey excision theorem implies that the diagram is $(r+s-1)$-cartesian. Since $r \geqslant 2$, we infer that the diagram is $(s+1)$-cartesian. Consequently, $X_\emptyset \to X_2$ is also $(s+1)$-connected. This completes the inductive step. □

COROLLARY 5.7. *Let*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

be a commutative diagram of connected spaces. Assume that

- *the outer square is j-cocartesian for some $j \geqslant 0$,*
- *the right-hand square is cocartesian, and*
- *$B \to C$ is 2-connected.*

Then the left-hand square is also j-cocartesian.

Proof. Assume without loss in generality that $A \to B$ and $B \to C$ are cofibrations. For formal reasons, if the right-hand square is cocartesian then so is the square

$$\begin{array}{ccc} X \cup_A B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X \cup_A C & \longrightarrow & Z \end{array} .$$

Since $B \to C$ is 2-connected we can apply Lemma 5.6(1) to infer that the left vertical map is also 2-connected. Now use Lemma 5.6(2). □

Notes 5.8. Richter [13] had the first proof of Theorem 5.1 under the assumption that all spaces of $X_\emptyset$ are simply connected (see Remark 5.2 for the proof in this instance). My original proof of the face lemma required each of the k_i to be $\geqslant 2$. The above proof, due to Goodwillie, places no constraints on the k_i.

6. PROOF OF THEOREM A

We recall the set-up of the introduction. Let K be a connected homotopy finite space with hodim$K \leqslant k$. Let $(X, \partial X)$ be a PD pair of dimension n. Let $f: K \to X$ be an r-connected map. Recall the statement of Theorem A:

THEOREM 6.1. *If $k \leqslant n-3$ and $r \geqslant 2k-n+2$, then f PD embeds.*

Proof. By Lemma 3.1 there exists a non-negative integer j such that the composite

$$K \xrightarrow{f} X \subset X \times D^j$$

PD embeds. By a downward induction on codimension, we may assume that $j = 1$. The strategy will be to recognize the PD embedding of $K \to X \times I$ as a decompression of a PD embedding of $f: K \to X$ (cf. Lemma 2.3).

Let

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & W \arrow[d] \\ K \arrow[r] & X \times I \end{tikzcd}$$

be a PD embedding of $K \xrightarrow{f} X \subset X \times I$ (in codimension $\geqslant n-k+1$). Recall that there is a factorization $\partial(X \times I) \to W \to X \times I$. The space $\partial(X \times I)$ is just $\Sigma_X \partial X$. In particular W is an object of $\mathbf{Top}_\nabla$ and $\Sigma_X \partial X \to W$ is a morphism of $\mathbf{Top}_\nabla$.

Using functorial factorization, we may assume that W is fibrant. Using the projection $X \times I \to X$, we will from now on be considering the square given by replacing $X \times I$ by X. We can also assume that the map $f: K \to X$ is a fibration.

CLAIM 6.2. *The object W desuspends relative to $\Sigma_X \partial X$, i.e., there is a cofibrant object $C \in \mathbf{Top}_{/X}$, a cofibration $\partial X \rightarrowtail C$ and a weak equivalence*

$$\Sigma_X C \xrightarrow{\sim} W$$

which is relative to $\Sigma_X \partial X$. Moreover, the map $C \to X$ can be chosen as $(n-k-1)$-connected.

The proof of Claim 6.2 will use the Desuspension Theorem 4.9. Since $W \to X \times I$ opposes $A' \to K$ in a cocartesian square, Lemma 5.6(1) shows that the object $W \in \mathbf{Top}_\nabla$ has connectivity one less than the connectivity of the map $A' \to K$. Since $(\bar{K}, A')$ is a PD pair of dimension $n+1$, this connectivity is just $n-k$. Consequently, W is an $(n-k-1)$-connected object. In particular, the codimension $\geqslant 3$ hypothesis says that X and W have isomorphic fundamental groups, so every local system on W arises by pullback from one on X.

Furthermore, there are isomorphisms

$$H^*(\bar{W}, \Sigma_X \partial X) \overset{\cap [W]}{\cong} H_{n+1-*}(\bar{W}, A') \overset{\text{excision}}{\cong} H_{n+1-*}(\bar{X}, K)$$

for any local system on X. Hence, the fact that f is r-connected implies these groups vanish whenever $* \geqslant n-r+1$. Therefore the map $\Sigma_X \partial X \to W$ has vanishing relative cohomology in these degrees.

Applying Addendum 4.9, we see that W desuspends relative to $\Sigma_X \partial X$ provided that $n-r \leqslant 2(n-k-1)+1$. This will happen if $r \geqslant 2k-n+1$, so we have one dimension to spare. This establishes Claim 6.2.

Let $K \amalg K \to K$ be the fold map. Let $K \amalg K \to W$ be the composite

$$K \amalg K \xrightarrow{f \amalg f} X \amalg X \subset W.$$

These maps are compatible with projection to X, and therefore define a map

$$K \amalg K \;\to\; K \times_X W,$$

where the target denotes the fiber product of K with W along X (recall we have arranged it so that $f\colon K \to X$ is a fibration, so the fiber product has the correct homotopy type). By the Blakers–Massey theorem, the map $A' \to K \times_X W$ is $(r+n-k-1)$-connected. But $K \amalg K$ has hodim $\leqslant k$, so obstruction theory gives us a factorization up to homotopy

$$K \amalg K \to A' \to K \times_X W$$

provided $r \geqslant 2k - n + 1$, so we again have one dimension to spare. By functorial factorization, we can assume that the map $A' \to K \times_X W$ is a fibration. But then the homotopy lifting property gives us a factorization on the nose.

The data constructed thus far may be displayed as the following commutative 3-dimensional punctured cube:

$$\begin{tikzcd}
 & & & K \arrow[dl] \arrow[dd] \\
K \arrow[rr] \arrow[dd] & & A' \arrow[dd] & \\
 & C \arrow[rr] \arrow[dl] & & \bar{X} \arrow[dl] \\
\bar{X} \arrow[rr] & & W &
\end{tikzcd}.$$

The bottom 2-face of this cube is the cocartesian square associated with the weak equivalence $\Sigma_X C \xrightarrow{\sim} W$; the space $\bar{X}$ denotes the mapping cylinder of $C \to X$.

Let B be homotopy inverse limit of the punctured cube. Then the resulting 3-cube of spaces

$$\begin{tikzcd}
 & B \arrow[rr] \arrow[dl] \arrow[dd] & & K \arrow[dl] \arrow[dd] \\
K \arrow[rr] \arrow[dd] & & A' \arrow[dd] & \\
 & C \arrow[rr] \arrow[dl] & & \bar{X} \arrow[dl] \\
\bar{X} \arrow[rr] & & W &
\end{tikzcd}$$

is commutative up to canonical homotopy.

It will be more convenient to work with a commutative version of this cube. One way to do this is as follows: map the original punctured cube to a new punctured cube by a pointwise weak equivalence, in such a way that the limit of the new punctured cube is the homotopy limit of the original punctured cube. The new punctured cube together with its limit gives the desired strictly commutative cube. In what follows, we will be working with the commutative cube. However, to avoid a notational clutter, we will keep the notation of the old cube to designate the spaces in the new one.

Consider next the top 2-face of the 3-cube.

Claim 6.3. *The top* 2-*face*

$$\begin{tikzcd} B \arrow[r] \arrow[d] & K \arrow[d] \\ K \arrow[r] & A' \end{tikzcd}$$

is $(2(n-k-1)+r)$-*cocartesian. Moreover, the space* B *is connected.*

We wish to apply the Face Theorem 5.1. We therefore need to verify its hypotheses.

All spaces of the 3-cube with the exception of perhaps B are connected. It is straightforward to check that each 2-face meeting W is cocartesian. The maps labeled $K \to A'$ and $C \to X$ are $(n-k-1)$-connected. The maps labeled $K \to X$ are r-connected. With the notation as in the Face Theorem, this means $k_1 = k_3 = n-k-1$ and $k_2 = r$. Since $k \leqslant n-3$, we infer that $k_1, k_3 \geqslant 2$. Consequently, we may apply the Face Theorem to conclude that B is connected and that the square is $(2(n-k-1)+r)$-cocartesian. This proves Claim 6.3.

We continue to restrict our attention to the top face.

Claim 6.4. *There exists a connected space* A *and a* $(2(n-k-1)+r-2)$-*connected map* $A \to B$ *such that the square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & K \arrow[d] \\ K \arrow[r] & A' \end{tikzcd}$$

(*given by replacing* B *by* A), *is cocartesian.*

Choose a basepoint for B to equip the top 2-face with the structure of a square of based spaces. The map $K \vee K \to A'$ has vanishing relative cohomology (with respect to any local system on A') in degrees $\geqslant n$, since A' is a PD space of dimension n and $k \leqslant n-3$. Thus if

$$n \leqslant 2(n-k-1)+r,$$

i.e., when $r \geqslant 2k-n+2$, we can apply 4.2 to obtain a space A and a $(2(n-k-1)+r-2)$-connected map $A \to B$ which satisfies the statement of the claim.

We note that this is the first (and only) time in the argument that the sharp lower bound for the connectivity of f is used.

Now consider one of the other 2-faces of the 3-cube meeting B, labeled

$$\begin{tikzcd} B \arrow[r] \arrow[d] & C \arrow[d] \\ K \arrow[r] & \bar{X} \end{tikzcd} .$$

Recall that $\partial X \to X$ comes equipped with a factorization $\partial X \to C \to X$. Replace B by A, and replace $\bar{X}$ by X to obtain a new commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ K \arrow[r] & X \end{tikzcd}.$$

CLAIM 6.5. *This square is cocartesian.*

The see this, consider the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & \bar{X} \arrow[d] \\ K \arrow[r] & \bar{X} \arrow[r] & W \end{tikzcd}.$$

The right-hand square is clearly cocartesian. The outer one is also cocartesian because it factors as a pair of cocartesian squares

$$\begin{tikzcd} A \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & X \arrow[d] \\ K \arrow[r] & A' \arrow[r] & W \end{tikzcd}.$$

The map $C \to X$ is 2-connected by construction. Then Claim 6.5 follows by application of Corollary 5.7 to the previous diagram.

CLAIM 6.6. *For the cocartesian square of Claim* 6.5, *we have*

(1) *The map $A \to K$ is $(n-k-1)$-connected (in particular, it is 2-connected).*
(2) *The spaces A and C are homotopy finite.*
(3) *The pair $(\bar{K}, A)$ is a PD pair of dimension n with fundamental class induced from $[X]$.*

To prove (1), we return to the cocartesian square of Claim 6.4. The map $A' \to K$ which makes $(\bar{K}, A')$ a PD pair is $(n-k)$-connected (since it is part of a PD embedding). The maps $K \to A'$ of the square are coretractions to $A' \to K$. Hence, the maps $K \to A'$ are $(n-k-1)$-connected. Applying Lemma 5.6(2), we see that $A \to K$ is also $(n-k-1)$-connected, since $k \leqslant n-3$.

To prove that A is homotopy finite, recall that a connected based space Y is homotopy finite if and only if $\pi_1(Y)$ is finitely presented and the associated $\mathbb{Z}[\pi_1(Y)]$-module chain complex $C_*(Y)$ is chain homotopy finite in the sense that it is equivalent to a bounded above chain complex which is degreewise finitely generated and free (see [18, 2.2]).

Since $\pi_1(A) = \pi_1(K)$ and K is homotopy finite, we infer that $\pi_1(A)$ is finitely presented. Now use the homotopy cofiber sequence

$$C_*(A) \to C_*(K) \to C_*(A', K)$$

and the fact that A' and K are homotopy finite to conclude that $C_*(A)$ is chain homotopy finite. A similar argument shows that C is homotopy finite. This establishes (2).

Assertion (3) follows from the isomorphism

$$H^*(\bar{K}, A) \cong H^{*+1}(\bar{K}, A')$$

(induced by the cocartesian square of Claim 6.4 with respect to any coefficient system on K, using the fact that the fundamental class for $(\bar{K}, A')$ is induced from $[X \times I]$.

This completes the proof of Claim 6.6.

From the above it follows that

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ K \arrow[r] & X \end{tikzcd}$$

is a PD embedding. However, recall that we chose to replace the original homotopy commutative 3-cube by a strictly commutative one (cf. before Claim 6.3). In doing so, the space K got replaced by something else homotopy equivalent to it (although we did not change the notation). The proof of Theorem A is completed by invoking Lemma 2.5(2). $\square$

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