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ON TWO RESULTS ABOUT FIBRATIONS

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Using equivariant methods, we provide straightforward proofs of a result of Chachólski and a result of Spivak about fibrations.

INTRODUCTION

Let $\tilde{h}_*$ denote a reduced homology theory on the category of based spaces and let h_* denote the corresponding extension to an unreduced homology theory on the category of unbased spaces given by $h_*(X) = \tilde{h}_*(X_+)$, where X_+ denotes X with the addition of a disjoint basepoint.

The following was proved by W. Chachólski [2], using the elaborate machinery of 'cellular inequalities' developed by E. Dror Farjoun (see [3]):

Theorem A. *Suppose that*

$$F \to E \to B$$

is a fibration, with B a connected, based space. Assume that

$$h_*(F) \to h_*(E)$$

is an isomorphism for all degrees $*$. Then $\tilde{h}_*(B) = 0$.

Chachólski tells me that Dror Farjoun has asked if a 'classical proof' of Theorem A exists. The first part of this paper provides such a proof.

It is perhaps worth remarking here that if the fibration admits a section the result is trivial, since the natural map $E \cup CF \to B$ is then a retraction, and the mapping cone $E \cup CF$ has trivial homology. This gives a clue as to how the result is to be proved in the general case: in effect, we will show that a homological retraction still exists even when the fibration doesn't admit a section.

The second result we shall be concerned with is crucial for establishing the existence of the Spivak normal fibration for Poincaré duality spaces. It is originally due to M. Spivak [5,4.3]. We state the reformulation of it given by W. Browder [1,I.4.3].

Theorem B. *Suppose that*

$$F \to E \xrightarrow{p} B$$

is a fibration, where B is 1-connected. Suppose also that the Thom isomorphism is satisfied, i.e., for some positive integer $n \geq 2$ there exists a class $u \in H^n(p)$ in singular cohomology such that the induced homomorphism

$$u\cap : H_*(p) \to H_{*-n}(B)$$

given by cap product with u is an isomorphism for all $*$. Then F is a homology $(n-1)$-sphere. In particular, if F is 1-connected, then $F \simeq S^{n-1}$.

The Spivak-Browder proof of Theorem B was computational, making intricate use of the relative Serre spectral sequence.

In contrast, our methods of proving Theorems A and B will be to convert the statements into questions involving spectra equipped with the action of a topological group. It is well-known that a connected, based space B has the weak homotopy type of BG, where G denotes the geometric realization of the Kan loop group of its total singular complex. With respect to this identification, a fibration over B corresponds to the Borel construction on a space with G-action. This observation enables us to apply standard equivariant techniques.

A version of Theorem B is also true without the 1-connected hypothesis on the base (cf. Ranicki [4,3.10]). In this instance, the Thom isomorphism is to be formulated with twisted coefficients. The methods of this paper can be used to obtain the general case (see 4.2 below). However, we only present the proof in the case of a 1-connected base to economize on technicality.

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Outline of paper. §1 is preliminary. §2 contains the proof of Theorem A. §3 contains the proof of Theorem B. §4 is an addendum which contains variants of Theorems A and B which we state without proof.

1. Preliminaries

The spaces in this paper are to be given the compactly generated topology and products are to be taken in the compactly generated sense. To insure that some of our constructions are homotopy invariant, we will also generally assume that (based) spaces are (based) CW complexes (when a construction leads us out of this class of spaces, we implicitly apply the functor $X \mapsto |SinX|$, i.e.. the geometric realization functor composed with the total singular functor).

1.1. Spectra with G-action. Let $G.$ be a simplicial group. The geometric realization $G := |G.|$ of its underlying simplicial set is then a topological group.

A *pre-spectrum with G-action* $\mathbf{W}$ is a collection of based (left) G-spaces $\mathbf{W}_n$, $n \in \{0,1,...\}$, and equivariant maps $\Sigma\mathbf{W}_n \to \mathbf{W}_{n+1}$ where G acts trivially on the suspension coordinate in the source. We say that $\mathbf{W}$ is a *spectrum with G-action* when the adjoint map $\mathbf{W}_n \to \Omega \mathbf{W}_{n+1}$ is a weak homotopy equivalence (of underlying topological spaces).

Given a pre-spectrum with G-action $\mathbf{W}$, we can form its *spectrification* $\mathbf{W}^\sharp$ by taking $\mathbf{W}^\sharp_n$ to be the homotopy colimit

$$\operatorname{hocolim}_k \Omega^k \mathbf{W}_{n+k} .$$

This is a spectrum with G-action.

Given a based G-space Y, there is an associated spectrum with G-action $Y \wedge \mathbf{W}$: it is defined to be the spectrification of the pre-spectrum with G-action whose n-th space is $Y \wedge \mathbf{W}_n$, with G-acting diagonally.

If $\mathbf{W}$ is a pre-spectrum with G-action, its *homotopy orbit* spectrum $\mathbf{W}_{hG}$ is given by spectrifying the pre-spectrum with G-action whose n-th space is the reduced Borel construction on $\mathbf{W}_n$, i.e., $EG_+ \wedge_G \mathbf{W}_n$, where EG denotes a free contractible G-space.

By a map of (pre-)spectra $f \colon \mathbf{X} \to \mathbf{Y}$ with G-action, we mean a collection of G-maps $f_n \colon \mathbf{X}_n \to \mathbf{Y}_n$ which is compatible with the structure maps.

Given a (pre)-spectrum $\mathbf{X}$, recall that its *homotopy groups* $\pi_j(\mathbf{X})$ for $j \in \mathbb{Z}$ are given by $\operatorname{colim}_{j+n}(\mathbf{X}_n)$. We say $\mathbf{X}$ that *r-connected* if its homotopy groups vanish below degree r. We say that $\mathbf{X}$ is *bounded below* if it is r-connected for some integer r. If $\mathbf{X}$ is a spectrum with G-action, its homotopy groups are a left $\mathbb{Z}[\pi_0(G)]$-module. The action of $[g] \in \pi_0(G)$ on $\pi_*(\mathbf{X})$ is defined by applying homotopy groups to the multiplication by g map.

Call a map $\mathbf{X} \to \mathbf{Y}$ of (pre-)spectra (with or without G-action) a *weak equivalence* if the induced map on homotopy groups is an isomorphism in every degree. Note that the canonical map from a pre-spectrum to its spectrification is a weak equivalence.

We use the notation $\mathbf{S}^0$ for the *sphere spectrum*, whose n-th space is $Q(S^n)$ (where $Q = \Omega^\infty \Sigma^\infty$, denotes the representing functor for stable homotopy). We also let $\mathbf{h}\mathbb{Z}$ denote the *Eilenberg-Mac Lane spectrum* with n-th space $K(\mathbb{Z}, n)$ (the Eilenberg-Mac Lane space of type $(\mathbb{Z}, n)$).

The following lemma shows that the homotopy orbit construction on the category of pre-spectra with G-action is weak homotopy invariant. It is probably well-known.

Lemma 1.2. *(1). If $\mathbf{X} \to \mathbf{Y}$ is a weak equivalence of pre-spectra with G-action, then the induced map of spectra*

$$\mathbf{X}_{hG} \to \mathbf{Y}_{hG}$$

is a weak equivalence.

(2). More generally, if $\mathbf{X} \to \mathbf{Y}$ is j-connected, then so is $\mathbf{X}_{hG} \to \mathbf{Y}_{hG}$.

Proof. (1). If $\mathbf{X}$ and $\mathbf{Y}$ are both spectra with G-action, then the result is clear because a map $\mathbf{X} \to \mathbf{Y}$ is a weak equivalence in this case if and only if the maps $\mathbf{X}_n \to \mathbf{Y}_n$ are for every $n \geq 0$. Assume therefore that $\mathbf{X}$ is a pre-spectrum with G-action. By functoriality, it will suffice to show that the map $e_{\mathbf{X}} \colon \mathbf{X} \to \mathbf{X}^\sharp$ given by including a pre-spectrum into its spectrification induces a weak equivalence on homotopy orbits. In what follows below, we use the notation $EG_+ \wedge_G \mathbf{X}$ to denote the pre-spectrum whose n-th term is $EG_+ \wedge_G \mathbf{X}_n$ (which spectrifies to $\mathbf{X}_{hG}$).

There is a map of pre-spectra

$$EG_+ \wedge_G \mathbf{X}^\sharp \xrightarrow{q_\mathbf{X}} (EG_+ \wedge_G \mathbf{X})^\sharp$$

which is defined on n-th spaces to be the inclusion of the first term into the homotopy colimit of the chain of maps

$$EG_+ \wedge_G \mathbf{X}^\sharp_n \to \Omega(EG_+ \wedge_G \mathbf{X}^\sharp_{n+1}) \to \Omega^2(EG_+ \wedge_G \mathbf{X}^\sharp_{n+2}) \to \cdots .$$

The composite map of pre-spectra

$$EG_+ \wedge_G X \xrightarrow{\mathrm{id}_{EG_+} \wedge_G e_\mathbf{X}} EG_+ \wedge_G \mathbf{X}^\sharp \xrightarrow{q_\mathbf{X}} (EG_+ \wedge_G X)^\sharp$$

coincides with the evident map from the pre-spectrum $EG_+ \wedge_G X$ into its spectrification, and is therefore a weak homotopy equivalence. In particular, the map

$$EG_+ \wedge_G \mathbf{X} \xrightarrow{\mathrm{id}_{EG_+} \wedge_G e_\mathbf{X}} EG_+ \wedge_G \mathbf{X}^\sharp$$

induces an injection on homotopy groups in all degrees.

We next claim that the composition

$$EG_+ \wedge_G \mathbf{X}^\sharp \xrightarrow{q_\mathbf{X}} (EG_+ \wedge_G \mathbf{X})^\sharp \xrightarrow{(\mathrm{id}_{EG_+} \wedge_G e_\mathbf{X})^\sharp} (EG_+ \wedge_G \mathbf{X}^\sharp)^\sharp$$

coincides with the spectrification map $e_{EG_+ \wedge_G \mathbf{X}^\sharp}$ up to homotopy. To see why, observe that there is an evident map

$$EG_+ \wedge_G \operatorname{hocolim}_{i,j} \Omega^{i+j} \mathbf{X}_{n+i+j} \to (EG_+ \wedge_G \mathbf{X}^\sharp)^\sharp_n .$$

Composing the latter with each of the 'axial' inclusions into the displayed double homotopy colimit (the maps given by setting $i = 0$ or $j = 0$) defines the two maps in question. A homotopy between these inclusions is defined by taking a suitable rotation.

Therefore, the map $\mathrm{id}_{EG_+} \wedge_G e_\mathbf{X}$ also induces a surjection on homotopy groups in all degrees (since its spectrification $(\mathrm{id}_{EG_+} \wedge_G e_\mathbf{X})^\sharp$ does). Thus, the map $\mathrm{id}_{EG_+} \wedge_G e_\mathbf{X}$ is a weak equivalence, as was to be proved.

(2). By the first part, we can assume that $\mathbf{X}$ and $\mathbf{Y}$ are spectra with G-action. It follows that the map of n-th spaces $\mathbf{X}_n \to \mathbf{Y}_n$ is $(j+n)$-connected. Since the reduced Borel construction preserves connectivity, we infer that the map $EG_+ \wedge_G \mathbf{X}_n \to EG_+ \wedge_G \mathbf{Y}_n$ is also $(j+n)$-connected. It follows that $\mathbf{X}_{hG} \to \mathbf{Y}_{hG}$ is j-connected. □

Lemma 1.3. *Let* $\mathbf{X}$ *be bounded below pre-spectrum with* G*-action and assume that* G *is connected. Furthermore, suppose that* X_{hG} *has trivial homotopy groups. Then* $\mathbf{X}$ *has trivial homotopy groups.*

Proof. If $\mathbf{X}$ is bounded below, there exists an integer j such that $\mathbf{X}$ is $(j-1)$-connected. By induction, it will be sufficient to prove that $\mathbf{X}$ is also j-connected. We can by 1.2(1) assume without loss in generality that $\mathbf{X}$ is also spectrum with G-action.

The map of based G-spaces

$$G_+ \to S^0$$

is 1-connected, since G is connected. Hence, smashing with $\mathbf{X}$, it follows that the induced map of spectra with G-action

$$G_+ \wedge \mathbf{X} \to S^0 \wedge \mathbf{X} = \mathbf{X}$$

is $(j+1)$-connected. The induced map of homotopy orbit spectra is identified with the natural map

$$\mathbf{X} \to \mathbf{X}_{hG}\,,$$

so by 1.2(2), the latter map is also $(j+1)$-connected. Consequently, $\mathbf{X}$ is j-connected. □

The following is a partial converse to 1.2(1).

Corollary 1.4. *If* G *is connected, then a map* $\mathbf{X} \to \mathbf{Y}$ *of bounded below prespectra with* G*-action which induces a weak equivalence on homotopy orbits was also a weak equivalence to begin with.*

Proof. Let $\mathbf{C}$ denote the homotopy cofibre of the map $\mathbf{X} \to \mathbf{Y}$. Then $\mathbf{C}$ is bounded below, and $\mathbf{C}_{hG}$ has trivial homotopy groups. Applying 1.3, we find that $\mathbf{C}$ has trivial homotopy groups. Hence $\mathbf{X} \to \mathbf{Y}$ is a weak equivalence. □

1.5. Fibrations. Let B be a connected, based space. It is well known that B is functorially weak homotopy equivalent to the classifying space $BG := EG/G$ where $G = |G.|$ is the realization of a sutiable simplicial group. Namely, one defines $G.$ as the Kan loop group of the total singular complex of B. For this reason, we from now on assume that B is identified with BG.

The following lemma says that in the homotopy category of spaces over BG, every fibration over BG is isomorphic to a Borel construction.

Lemma 1.6. *Given a (Serre) fibration* $p\colon E \to B$ *with fiber* $F = p^{-1}(*)$*, there exists an unbased* G*-space* F' *and a weak homotopy equivalence*

$$E \to EG \times_G F'$$

compatible with projection to $B = BG$*, where the target denotes the unreduced Borel construction of* G *acting on* F'*. Moreover, the induced map* $F \xrightarrow{\simeq} F'$ *is a weak homotopy equivalence of underlying spaces.*

Proof. Form the pullback of p along the universal G-fibration $EG \to BG = B$. Let $E^* \to EG$ denote the resulting fibration. Then the fiber of the latter over the basepoint is F, and since EG is contractible, $F \to E^*$ is a weak homotopy equivalence. Moreover E^* comes equipped with the structure of a G-space, and the orbit space E^*/G is identified with E in a way such that $E^*/G \to EG/G$ coincides with the projection $E \to B$. On the other hand, the action of G on E^* is free, and the natural map $E^*/G \to EG \times_G E^*$ is a weak homotopy equivalence which commutes with projection to BG. Hence we may define $F' := E^*$. □

By means of 1.6, we may assume that any fibration $E \to B$ is a Borel construction with $B = BG$.

2. Proof of Theorem A

Let
$$\tilde{h}_*\colon \mathbf{Top}_* \to \mathbf{Ab}$$
be a reduced homology theory on the category of based spaces. Then h_* will denote the corresponding unreduced homology theory on the category of unbased spaces.

By the Brown representability theorem, there exists a spectrum $\mathbf{W}$ such that $\tilde{h}_*(Y) = \pi_*(\mathbf{W} \wedge Y)$ up to natural isomorphism. Using $\mathbf{W}$ we obtain a functor $h\colon \mathbf{Top} \to \mathbf{Sp}$ from unbased spaces to spectra by taking $Z \mapsto \mathbf{W} \wedge Z_+$. By taking homotopy groups, this recovers the unreduced theory h_*.

Given a fibration
$$F \xrightarrow{i} E \xrightarrow{p} B$$
satisfying the hypotheses of Theorem A, we may assume by 1.6 that $B = BG$, F is a G-space and $E = EG \times_G F$ is a Borel construction.

Let $\mathbf{W}$ represent h_*. Define a left G-action on the spectrum $h(F) = \mathbf{W} \wedge F_+$ by letting G act trivially on $\mathbf{W}$ and taking the resulting diagonal action on the smash product.

Lemma 2.1. *Up to weak equivalence, $h(E)$ is given by the homotopy orbit spectrum*
$$h(F)_{hG}\,.$$
Moreover, with respect to this identification, the map $h(i)\colon h(F) \to h(E)$ coincides with the natural map $h(F) \to h(F)_{hG}$ from a spectrum with G-action to its homotopy orbit spectrum.

Proof.
$$\begin{aligned} h(E) &= h(EG \times_G F) \\ &= \mathbf{W} \wedge (EG \times_G F)_+ \\ &= \mathbf{W} \wedge (EG_+ \wedge_G F_+) \\ &\simeq (\mathbf{W} \wedge F_+)_{hG} \\ &= h(F)_{hG} \end{aligned}$$

The establishes the first part. The second part is just a short diagram chase. □

Proof of Theorem A. By 2.1, the map
$$h(F) \to h(E)$$

is identified up to homotopy with the natural map

$$h(F) \to h(F)_{hG}.$$

By hypothesis, it is a weak equivalence. Moreover, it is G-equivariant, where the action on the target is defined to be trivial. Let us rename the target spectrum by $\mathbf{V}$. Then the map $h(F) \to \mathbf{V}$ is a weak equivalence of spectra with G-action, and therefore

$$h(F)_{hG} \simeq \mathbf{V}_{hG} = BG_+ \wedge \mathbf{V} = B_+ \wedge \mathbf{V} \simeq B_+ \wedge h(F),$$

where the first equality uses the fact that G acts trivially on $\mathbf{V}$. With respect to this chain of equivalences, the weak equivalence $h(F) \to h(F)_{hG}$ may be identified with the inclusion map

$$h(F) \to B_+ \wedge h(F).$$

Now, using the fact $h(F) = \mathbf{W} \wedge F_+$, we see that

$$B_+ \wedge h(F) = B_+ \wedge (\mathbf{W} \wedge F_+) \simeq \mathbf{W} \wedge (B_+ \wedge F_+) = \mathbf{W} \wedge (B \times F)_+ = h(B \times F).$$

We therefore have that $h(E) \simeq h(B \times F)$ and moreover, modulo this identification, the map $h(F) \to h(E)$ is given by applying h to the inclusion $F \to B \times F$.

Consequently, the hypotheses imply that the map

$$h(F) \to h(B \times F)$$

is a weak equivalence. Let $* \in F$ be a choice of basepoint. Then the map $* \to B \times *$ is a retract of the inclusion $F \to B \times F$. Hence $h(*) \to h(B)$ is also a weak equivalence. This means that the reduced homology of B, i.e., $\tilde{h}_*(B)$, is trivial. This completes the proof of Theorem A. □

3. Proof of Theorem B

In order to prove Theorem B, it will be helpful to reformulate the Thom isomorphism for fibrations equipped with a section.

Let $F \to E \xrightarrow{p} B$ be a fibration equipped with a section $s\colon B \to E$ such that the section is a cofibration.

Define a diagonal map

$$\Delta\colon E/B \to (E/B) \wedge B_+$$

by taking induced quotients on the map of pairs $(E, B) \to (E \times B, B \times B)$ defined by $e \mapsto (e, p(e))$.

Then the cofibration sequence

$$E \cup_s C(B) \to E \cup_{\mathrm{id}} C(E) \to B \cup_p C(E)$$

shows that $\Sigma(E/B)$ is weak equivalent to $B \cup_p C(E)$. Consequently, there is an isomorphism $H^n(E \cup_s C(B)) \cong H^{n+1}(B \cup_p C(E))$.

Example 3.1. Let $F_0 \to E_0 \xrightarrow{p_0} B$ be a fibration. Taking the fiberwise join with S^0, we obtain a fibration $\Sigma F_0 \to E \to B$ with section $B \to E$. In this instance, we have a weak equivalence

$$E/B \quad \simeq \quad B \cup_{p_0} C(E_0)\,.$$

The latter is often called the 'Thom complex' of the fibration $E_0 \to B$.

Definition 3.2. Let $\tilde{H}^*$ be reduced singular cohomology. We say that p has a *Thom class* (of degree n), if it is equipped with a class $u \in \tilde{H}^n(E/B)$ such that capping with u by means of the diagonal map Δ gives an isomorphism in singular homology

$$u\cap\colon \tilde{H}_*(E/B) \quad \xrightarrow{\cong} \quad \tilde{H}_{*-n}(B_+)$$

for all degrees $*$.

Let $\tilde{H}\colon \mathbf{Top}_* \to \mathbf{Sp}$ represent reduced singular homology, i.e.,

$$\tilde{H}(Y) \quad = \quad \mathbf{h}(\mathbb{Z}) \wedge Y$$

where we recall that $\mathbf{h}(\mathbb{Z})$ denotes the Eilenberg-Mac Lane spectrum.

Lemma 3.3. *Suppose that p has a Thom class of degree n. Then there is a G-equivariant map*

$$\tilde{H}(F) \to \tilde{H}(S^n)$$

where the target has the trivial G-action. Moreover, this map yields a weak equivalence upon taking homotopy orbits.

Proof. Using the fact $\tilde{H}^n(E/B) = [E/B, \mathbf{h}(\mathbb{Z})_n]$, we may represent the Thom class as a map of spectra

$$\mathrm{S}^0 \wedge E/B \xrightarrow{u} S^n \wedge \mathbf{h}(\mathbb{Z})\,.$$

Smash this on the right with $\mathbf{h}(\mathbb{Z})$ and then compose with the multiplication map $\mu\colon \mathbf{h}(\mathbb{Z})^{\wedge 2} \to \mathbf{h}(\mathbb{Z})$ to give a map

$$E/B \wedge \mathbf{h}(\mathbb{Z}) \simeq \mathrm{S}^0 \wedge E/B \wedge \mathbf{h}(\mathbb{Z}) \xrightarrow{u\wedge \mathrm{id}} S^n \wedge \mathbf{h}(\mathbb{Z}) \wedge \mathbf{h}(\mathbb{Z}) \xrightarrow{\mathrm{id}\wedge\mu} S^n \wedge \mathbf{h}(\mathbb{Z})\,.$$

Using the identification

$$F_{hG} \quad = \quad E/B$$

we therefore have a map

$$F_{hG} \wedge \mathbf{h}(\mathbb{Z}) \to S^n \wedge \mathbf{h}(\mathbb{Z}),$$

or equivalently, by giving $S^n \wedge \mathbf{h}(\mathbb{Z})$ the trivial G-action, an equivariant map

$$\tilde{H}(F) = F \wedge \mathbf{h}(\mathbb{Z}) \to S^n \wedge \mathbf{h}(\mathbb{Z}) = \tilde{H}(S^n)\,.$$

Finally, applying the homotopy orbit construction yields a map

$$E/B \wedge \mathbf{h}(\mathbb{Z}) = (F \wedge \mathbf{h}(\mathbb{Z}))_{hG} \quad \to \quad (S^n \wedge \mathbf{h}(\mathbb{Z}))_{hG} = B_+ \wedge S^n \wedge \mathbf{h}(\mathbb{Z}).$$

On homotopy groups, it is straightforward to check that the latter yields the cap product

$$u\cap \colon \tilde{H}_*(E/B) \to \tilde{H}_{*-n}(B_+).$$

By assumption, the latter is an isomorphism for all $*$. Consequently, the map

$$\tilde{H}(F)_{hG} \to \tilde{H}(S^n)_{hG}$$

is a weak homotopy equivalence. □

Proof of Theorem B. Let $F \to E \to B$ satisfy the Thom isomorphism, with B 1-connected. By taking fiberwise join with S^0, we see that the Thom isomorphism is also satisfied for the resulting fibration. We may therefore assume without loss in generality that the original fibration $E \to B$ is equipped with a section which is also a cofibration.

By 3.3, there exists a a G-equivariant map

$$\tilde{H}(F) \to \tilde{H}(S^n)$$

which induces a weak equivalence on homotopy orbits. Since $B = BG$ is 1-connected, G is connected. Therefore, applying 1.4 we conclude that the map $\tilde{H}(F) \to \tilde{H}(S^n)$ is a weak equivalence. Hence F is a homology n-sphere. □

4. Addendum

Using the methods of this paper, it is possible without much trouble to vary Theorems A and B in several directions. We will state these results without proof. In what follows, let

$$F \xrightarrow{i} E \xrightarrow{p} B$$

be a fibration with B a connected, based space.

Suppose that ξ denotes a locally trivial twisted coefficient system of abelian groups on B. Let $H_*(B;\xi)$ ($\tilde{H}_*(B;\xi)$) denote the (resp. reduced) homology of B with coefficients in ξ. Let $p^*\xi$ denote the restriction of ξ to E; note that the further restriction to F is constant: let this module be denoted by ξ_0.

The following is a variant of Theorem A:

Assertion 4.1. *Suppose that the map*

$$H_q(F;\xi_0) \to H_q(E;p^*\xi)$$

is an isomorphism for all q. Then $\tilde{H}_q(B;\xi)$ is trivial for all q.

This is a special case of a more general result concerning homology with coefficients in a 'bundle of spectra' parametrized by points of B (however, we won't bother to define what this means).

We now give a generalization of Theorem B. Suppose that $\mathbf{E}$ denotes a bounded below ring spectrum and let E_* and E^* denote the corresponding homology and cohomology theories. Suppose that the fibration satisfies the Thom isomorphism with respect to $\mathbf{E}$, i.e., there exists a class $u \in E^n(p)$ such that capping with u defines an isomorphism $E_*(p) \cong E_{n-*}(B)$.

Assertion 4.2. *With respect to these assumptions, it follows that F is a E_*-homology $(n-1)$-sphere.*

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