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Topology and its Applications 77 (1997) 1–18

TOPOLOGY AND ITS APPLICATIONS

Comultiplication and suspension

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Received 8 August 1995; revised 26 February 1996, 24 July 1996

Abstract

We prove the conjecture of Berstein, Hilton, and Ganea that an $(n-1)$-connected CW-complex X with $\dim X \leqslant k(n-2)+3$ which has a co-A_{k-1}-space structure desuspends. Moreover, we show the following dual of a result of Segal: given a cosimplicial space Y_* which is special in the sense that the canonical maps $\bigvee_{k=1}^{n} Y_1 \to Y_n$ are homotopy equivalences and Y_1 is 2-connected then there is a functorial desuspension of Y_1. © 1997 Elsevier Science B.V.

Keywords: Co-A_k-space; Desuspension; Cosimplicial space; Homotopy limit

AMS classification: 55P45; 55P40

1. Introduction

It has been known for a long time that an $(n-1)$-connected CW-complex of dimension $\leqslant 2n-1$ has the homotopy type of a suspension. In 1963 Berstein and Hilton proved that an $(n-1)$-connected based CW-complex X of dimension $\leqslant 3n-3$ has the homotopy type of a suspension provided X admits a comultiplication $X \to X \vee X$ with homotopy counit [1] (Berstein and Hilton made the additional technical requirement that the homology groups of X are finitely generated). In 1970 Ganea extended this result: an $(n-1)$-connected co-H-space X of dimension $\leqslant 4n-5$, $n \geqslant 2$, has the homotopy type of a suspension ΣY if and only if it is homotopy coassociative. Moreover, the homotopy equivalence $\Sigma Y \to X$ is a homomorphism up to homotopy [5]. Berstein–Hilton and Ganea conjectured that a suitable theory of co-A_n-spaces, dual to Stasheff's theory of A_n-H-spaces [13], would lead to extensions of their results. Saito has developed such a theory in part and managed to extend the results to co-A_4-spaces of dimension $\leqslant 5n-7$ [11]. In Stasheff's A_n-space situation the following result holds [14,17].

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PII S0166-8641(96)00103-4

Proposition 1.1. *Let X be an $(n-1)$-connected well-pointed A_{k-1}-space, $k \geqslant 3$, admitting a homotopy inverse, such that $\pi_i X = 0$ for $i > k(n+1) - 4$. Then there is an A_{k-1}-homomorphism $f: X \to \Omega_M Y$ which is a weak equivalence. Here $\Omega_M Y$ is the Moore loop space on Y with its natural monoid structure.*

Recall that an A_k-homomorphism is a homomorphism up to homotopy satisfying coherence conditions up to $(k-1)$-dimensional homotopies. The purpose of this paper is to prove the following dual result.

Theorem 1.2. *Let X be an $(n-1)$-connected based CW-complex with $\dim X \leqslant k(n-2)+3$, $n \geqslant 2$. Then X is of the homotopy type of a suspension if and only if X is a co-A_{k-1}-space. Moreover, the homotopy equivalence $f: \Sigma Y \to X$ can be chosen to be a co-A_{k-1}-homomorphism.*

Corollary 1.3. *A 2-connected space X is of the weak homotopy type of a suspension if and only if it is a co-A_∞-space.*

We then use our approach to prove a dual version of Segal's delooping result [12]

Theorem 1.4. *Let $X: \triangle \to \mathcal{T}op^*$ be a cosimplicial based space such that X_0 is contractible, X_1 is 2-connected and $(\pi_1, \ldots, \pi_n): \bigvee_{k=1}^n X_1 \to X_n$ is a homotopy equivalence where $\pi_i: [1] \to [n]$ maps 0 to $i-1$ and 1 to i. Then there is a weak homotopy equivalence $\Sigma RX \to X_1$, where RX is a functorial, thickened version of the topological corealization of X.*

A result of this type has been formulated by Hopkins [7] with the weaker condition that X_1 is only 1-connected. He claimed that the result follows from the homology spectral sequence of the cosimplicial space. In his analysis of this spectral sequence Bousfield addresses Hopkins result [3, (4.9)] but he requires that the corealization of X is nilpotent. This is certainly the case if X_1 is 2-connected, and we cannot see that 1-connectedness suffices. Instead of the maps π_k Hopkins uses the equivalence

$$(\iota_1, \ldots, \iota_n): \bigvee_{k=1}^n X_1 \xrightarrow{\simeq} X_n$$

where $\iota_k: [1] \to [n]$ maps 0 to 0 and 1 to k. A cosimplicial space in his sense induces one in our sense by a base change.

The approach to desuspension which we develop here is motivated by potential applications. A fibrant variant of the main result of this paper (see [9]) will be used by the first-named author in a subsequent paper to prove a Haefliger-type embedding theorem in the category of Poincaré duality spaces [8]. Such embedding and desuspension results also play a fundamental role in the original Poincaré surgery program as formulated by Browder et al. It is hoped that the results of this paper can be adapted so as to complete the Browder program.

While Ganea and Saito make use of the comonad $\Sigma\Omega$ in their proofs our proof is more in the spirit of Berstein–Hilton who construct the desuspension directly, although our methods differ. They carefully analyze the homology of the spaces involved (which requires the assumption that these groups are finitely generated) while we use the homotopy theory of cubical diagrams developed by Goodwillie [6]. To stay away from point set topological difficulties we work in the category $\mathcal{T}op$ of compactly generated spaces in the sense of [15]. Based spaces are assumed to be well-pointed unless stated otherwise.

2. A_k-spaces and co-A_k-spaces

We recall the definition of an A_k-space using the terminology of [2].

Let $\mathcal{G}$ be the category with $ob\,\mathcal{G} = \mathbb{N} = \{0, 1, \ldots\}$, exactly one morphism $\lambda_n : n \to 1$ for $n \geqslant 1$, and $\mathcal{G}(n, k)$ consisting of all formal expressions

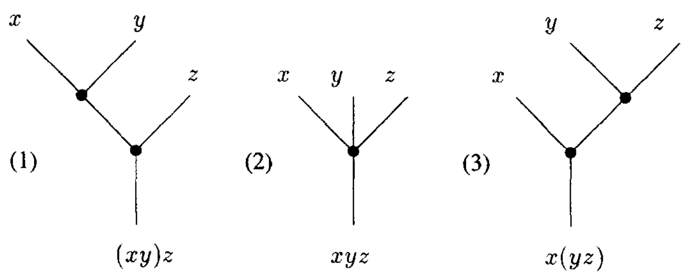
$$\lambda_{i_1} \oplus \cdots \oplus \lambda_{i_k}, \quad i_1 + \cdots + i_k = n$$

$\mathcal{G}(0,0) = \{\mathrm{id}_0\}$ and $\mathcal{G}(0,n) = \emptyset = \mathcal{G}(n,0)$ for $n > 0$. Composition is defined by the requirement that $\oplus$ is a bifunctor.

The category $\mathcal{G}$ describes semigroups: a semigroup G determines a functor $\mathcal{G} \to \mathcal{T}op$, $n \mapsto G^n$, transforming $\oplus$ into $\times$. Conversely, if $G : \mathcal{G} \to \mathcal{T}op$ is a functor such that $G(n) = G(1)^n$ and $G(f \oplus g) = G(f) \times G(g)$ then $G(1)$ is a semigroup with multiplication $G(\lambda_2)$.

We symbolize λ_n, $n \geqslant 2$, by a box with n inputs $x_1, \ldots, x_n$ and one output, the product $x_1 \cdots x_n$. Composite operations are obtained by wiring boxes together to a tree shaped circuit, e.g.,

Example 2.1.



represent all possible nontrivial composite operations $3 \to 1$. In $\mathcal{G}$ they coincide but for homotopy associative multiplications they only coincide up to homotopy. To account for this we give each connection (between boxes) a length $t \in I = [0, 1]$. Given a tree T the various lengths of its connections make up the points of a cube $C(T)$ whose dimension is the number of connections of T. We thus obtain a new category $\mathcal{TG}$ with ob $\mathcal{TG}$ = ob $\mathcal{G}$ and $\mathcal{TG}(n, 1) = \coprod C(T)$, T running through all trees with n inputs,

$$\mathcal{TG}(n,k) = \coprod \mathcal{TG}(i_1, 1) \times \cdots \times \mathcal{TG}(i_k, 1) \quad \text{with } i_1 + \cdots + i_k = n.$$

Let T be a tree with lengths in $\mathcal{TG}(k,1)$ and let $(T_1,\dots,T_k)\in\mathcal{TG}(n,k)$ be a k-tuple of trees; then $T\circ(T_1,\dots,T_k)$ is the tree obtained from T by sticking T_i onto its ith input and giving the newly created connections the lengths 1. The trivial tree serves as id_1. We have a continuous pairing $\oplus:\mathcal{TG}\times\mathcal{TG}\to\mathcal{TG}$ defined by

$$(T_1,\dots,T_k)\oplus(T_{k+1},\dots,T_{k+l})=(T_1,\dots,T_{k+l}).$$

Composition then is determined by the requirement that $\oplus$ is a bifunctor. It is continuous.

Let $\mathcal{WG}$ be the quotient category of $\mathcal{TG}$ obtained by imposing the relation that a connection of length 0 may be shrunk by amalgamating the two boxes at its ends to a single one. Observe that the trees in Example 2.1(1) and (3) represent unit intervals in $\mathcal{WG}(3,1)$ which are identified at 0 with the tree (2). Hence the trees of Example 2.1 codify an associating homotopy.

Definition 2.2. Let $\mathcal{W}_k\mathcal{G}$ be the full subcategory of $\mathcal{WG}$ of objects $0,1,\dots,k$. An *A_k-space* is a continuous functor $X:\mathcal{W}_k\mathcal{G}\to\mathcal{T}op$ mapping $\oplus$ to $\times$ (in particular $X(n)=X(1)^n$) such that the multiplication $X(\lambda_2):X(1)^2\to X(1)$ admits a homotopy unit. (Note that we do not require any coherence for the homotopy unit.) By abuse of notation we often denote $X(1)$ by X.

Remark 2.3. The map $\varepsilon:\mathcal{WG}\to\mathcal{G}$ which maps a tree with n inputs to λ_n is a continuous functor admitting a nonfunctorial section $\eta:\mathcal{G}\to\mathcal{WG}$ mapping λ_n to the tree with a single vertex and n inputs if $n>1$ and to the trivial tree for $n=1$. As maps of morphism spaces $\eta\circ\varepsilon\simeq\mathrm{id}$ (the homotopy shrinks the lengths of connections by the factor t at time t).

It can be shown that $\mathcal{WG}(n,1)$ is a subdivision of the Stasheff cell K_n of [13] into cubes. So our definition coincides with the one of Stasheff with the exception that he makes the stronger requirement that $X(\lambda_2)$ admits a strict unit e. But if $\{e\}\subset X$ is a closed cofibration both definitions are equivalent [2, Chapter I].

Co-A_k-spaces are defined dually:

Definition 2.4. A *co-A_k-space* is a continuous functor $X:(\mathcal{W}_k\mathcal{G})^{\mathrm{op}}\to\mathcal{T}op^*$ into the category of well-based topological spaces mapping $\oplus$ to the wedge sum $\vee$ such that the comultiplication $X(\lambda_2):X(1)\to X(1)\vee X(1)$ admits a homotopy counit (i.e., $p_i\circ X(\lambda_2)\simeq\mathrm{id}$, where $p_i:X(1)\vee X(1)\to X(1)$ is the ith projection, $i=1,2$). Again we often write X for $X(1)$.

Example 2.5. Let C be an operad without permutations in the sense of [10]. We associate to C a category $\mathcal{C}$ with ob $\mathcal{C}=\mathbb{N}$,

$$\mathcal{C}(n,k)=\coprod C(i_1)\times\cdots\times C(i_k)\quad\text{with } i_1+\cdots+i_k=n \text{ and all } i_j>0.$$

$\mathcal{C}(0,0)=\{\mathrm{id}_0\}$ and $\mathcal{C}(0,n)=\mathcal{C}(n,0)=\emptyset$ if $n>0$. Taking products defines a pairing $\oplus:\mathcal{C}\times\mathcal{C}\to\mathcal{C}$. Composition

$$\mathcal{C}(k,1)\times\mathcal{C}(n,k)\to\mathcal{C}(n,1)$$

is defined by the structure map γ of the operad. General composition then is determined by the requirement that $\oplus$ is a bifunctor.

Call C an *A_k-operad* if $C(j) \simeq *$ for $j \leqslant k$. If C is an A_k-operad, there is a continuous $\oplus$-preserving functor $F_C : \mathcal{W}_k\mathcal{G} \to \mathcal{C}$ by [2, 3.17]. If F'_C is another such functor then there is a homotopy through continuous $\oplus$-preserving functors from F_C to F'_C.

A *co-C-space* is a collection of maps

$$\mu_n : X \wedge C(n)^+ \longrightarrow X \vee \cdots \vee X = X^{\vee n}$$

(n-fold wedge) where $C(n)^+ = C(n) \sqcup \{*\}$, such that $\mu_1(x,1) = x$ and

$$\begin{tikzcd}
X \wedge C(k)^+ \wedge C(i_1)^+ \wedge \cdots \wedge C(i_k)^+ \arrow[r, "\mu_k \wedge \mathrm{id}"] \arrow[d, "\cong"] & X^{\vee k} \wedge C(i_1)^+ \wedge \cdots \wedge C(i_k)^+ \arrow[d, "\mathrm{proj}"] \\
X \wedge (C(k) \times C(i_1) \times \cdots \times C(i_k))^+ \arrow[d, "\mathrm{id}\wedge\gamma"] & (X \wedge C(i_1)^+) \vee \cdots \vee (X \wedge C(i_k)^+) \arrow[d, "\mu_{i_1} \vee \cdots \vee \mu_{i_k}"] \\
X \wedge C(n)^+ \arrow[r, "\mu_n"] & X^{\vee n}
\end{tikzcd}$$

commutes. It determines a *co-$\mathcal{C}$-space*, i.e., a continuous functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{T}op^*$ mapping $\oplus$ to $\vee$, hence a continuous functor $(\mathcal{W}_k\mathcal{G})^{\mathrm{op}} \to \mathcal{T}op^*$ mapping $\oplus$ to $\vee$.

If C is an A_k-operad, $k \geqslant 2$, then any co-C-space has a homotopy counit. Hence it is a co-A_k-space.

Example 2.6. Let $\mathcal{Q}$ be the "little 1-cubes" category [2, 2.53] describing loop space structures. A point in $\mathcal{Q}(n,1)$ is a tuple $(x_1, y_1, \ldots, x_n, y_n)$ of points in I such that $0 \leqslant x_1 < y_1 \leqslant x_2 < y_2 \leqslant \cdots \leqslant x_n < y_n \leqslant 1$, and should be considered as an embedding of n intervals $[x_i, y_i]$ into I such that the images have disjoint interiors. The spaces $\mathcal{Q}(n,1)$ and the composition in $\mathcal{Q}$ define the little cube operad C_1 of [10].

Let Z be a based space. Then ΣZ is a co-$\mathcal{Q}$-space and hence a co-A_∞-space with

$$\mu_n : \Sigma Z \wedge \mathcal{Q}(n,1)^+ \to \Sigma Z \vee \cdots \vee \Sigma Z$$

$$\mu_n\big((z,t),(x_1,y_1,\ldots,x_n,y_n)\big) = \begin{cases} \left(z, \dfrac{t - x_k}{y_k - x_k}\right) \in k\text{th summand} \\ \qquad\qquad \text{if } t \in [x_k, y_k], \\ * \qquad\qquad \text{otherwise.} \end{cases}$$

Remark 2.7. In their treatment of multiplications on topological spaces Boardman and Vogt [2] and May [10] have homotopy units incorporated into the structure as 0-ary operations or points in the 0th space $C(0)$ of an operad C respectively. We here merely require homotopy units outside the structure, i.e., their existence without specifying a particular one. This weaker condition suffices for our purposes and simplifies our argument.

3. Cubical diagrams

For the reader's convenience we recall Goodwillie's results about cubical diagrams.

Let $\mathcal{Q}_n$ denote the category of all subsets of $\underline{n} = \{1, \dots, n\}$ and inclusions, and $\mathcal{J}_n$ and $\mathcal{T}_n$ the full subcategories of all objects except for $\underline{n}$ respectively $\emptyset$. For $n = 0$ we take $\underline{n} = \emptyset$. A diagram $D\colon \mathcal{Q}_n \to \mathcal{T}op^*$ is a cubical diagram of based spaces with $D(T)$ sitting at the corner with coordinates $(\varepsilon_1, \dots, \varepsilon_n)$ where

$$\varepsilon_i = \begin{cases} 1 & \text{if } i \in T, \\ 0 & \text{if } i \notin T, \end{cases}$$

$D|\mathcal{J}_n$ and $D|\mathcal{T}_n$ are the subdiagrams of all proper initial and terminal faces respectively.

Definition 3.1. D is called k-Cartesian if the canonical map $D(\emptyset) \to \operatorname{holim} D|\mathcal{T}_n$ is k-connected, and k-co-Cartesian if the canonical map $\operatorname{hocolim} D|\mathcal{J}_n \to D(\underline{n})$ is k-connected.

For $T \subset \underline{n}$ let $\partial^T D$ be the restriction of D to the $|T|$-dimensional face with initial vertex $\emptyset$ and terminal vertex T and $\partial_{\underline{n}-T} D$ the restriction of D to the $|T|$-dimensional face with initial vertex $\underline{n} - T$ and terminal vertex $\underline{n}$ ($|T|$ denotes the number of elements of T).

Proposition 3.2 (Goodwillie [6]). *Let $D\colon \mathcal{Q}_n \to \mathcal{T}op^*$, $n \geqslant 1$, be a diagram such that*

(i) *$\partial^T D$ is $c(T)$-co-Cartesian for all $T \subset \underline{n}$, $T \neq \emptyset$,*

(ii) *$c(U) \leqslant c(T)$ for $U \subset T$.*

Then D is k-Cartesian with

$$k = \operatorname{Min}\left\{1 - n + \sum_\alpha c(T_\alpha);\ \{T_\alpha\} \textit{ a partition of } \underline{n} \textit{ into } T_\alpha \neq \emptyset\right\}.$$

Proposition 3.3 (Goodwillie [6]). *Let $D\colon \mathcal{Q}_n \to \mathcal{T}op^*$, $n \geqslant 1$, be a diagram such that*

(i) *$\partial_{\underline{n}-T} D$ is $k(T)$-Cartesian for all $T \subset \underline{n}$, $T \neq \emptyset$,*

(ii) *$k(U) \leqslant k(T)$ for $U \subset T$.*

Then D is c-co-Cartesian with

$$c = \operatorname{Min}\left\{n - 1 + \sum_\alpha k(T_\alpha);\ \{T_\alpha\} \textit{ is a partition of } \underline{n} \textit{ into } T_\alpha \neq \emptyset\right\}.$$

Remark 3.4. We will have to apply these results to coherently homotopy commutative diagrams in the sense of Section 5 rather than commutative ones. Fortunately they hold in this case too (e.g., a stronger version of [6, Theorem 2.4] for homotopy commutative cube diagrams can be found in [18, Proposition 5.5]). Indeed, by [16, Proposition 5.4] each coherently homotopy commutative cube is equivalent to a strictly commutative one, and the equivalence induces an equivalence of their homotopy limits and colimits.

4. The Berstein–Hilton result

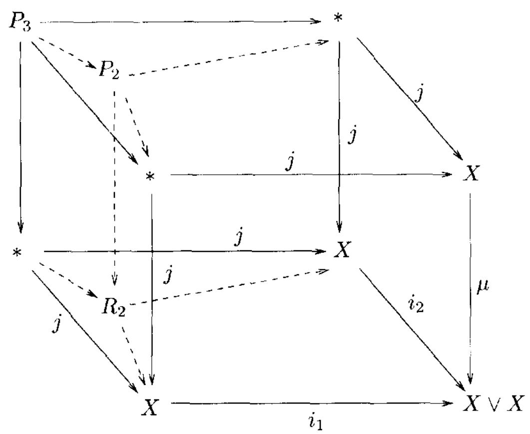
Before we prove Theorem 1.2 we illustrate our strategy by reproving the Berstein–Hilton result.

Let X be an $(n-1)$-connected co-H-space, $n \geqslant 2$, with comultiplication

$$\mu : X \to X \vee X.$$

Consider the diagram of based spaces (translating into cube coordinates $P_3 = D(\emptyset)$ corresponds to $(0,0,0)$ and $X \vee X = D(\underline{3})$ corresponds to $(1,1,1)$):

Diagram 4.1.



where i_k is the inclusion of the kth wedge summand. The maps i_1, i_2, μ and j define a diagram $\mathcal{T}_3 \to \mathcal{T}op^*$, which we complete to a $\mathcal{Q}_3$-diagram by taking the homotopy limit P_3. Note that P_3 can be obtained by taking iterated homotopy pullbacks: let R_2 be the homotopy pullback of

$$X \xrightarrow{i_1} X \vee X \xleftarrow{i_2} X,$$

and P_2 the one of $* \to X \leftarrow *$ (hence P_2 is the loop space ΩX and R_2 the cojoin $X \overset{\wedge}{*} X$); then P_3 is the homotopy pullback:

Diagram 4.2.

$$\begin{tikzcd} P_3 \arrow[r] \arrow[d] & P_2 \arrow[d] \\ * \arrow[r, "j'"] & R_2 \end{tikzcd}$$

The bottom face of Diagram 4.1 is a diagram from $\mathcal{Q}_2$ to $\mathcal{T}op^*$ which is co-Cartesian (meaning ∞-co-Cartesian) and j is $(n-1)$-connected. By Proposition 3.2 the map

$j' : * \to R_2$ and hence R_2 is $(2n-3)$-connected. Since Diagram 4.2 is a homotopy pullback diagram the map $P_3 \to P_2$ is also $(2n-3)$-connected. In particular, P_3 is connected.

Let M_3 be the homotopy colimit of Diagram 4.1 restricted to $\mathcal{J}_3$. We can obtain M_3 by taking iterated homotopy pushouts: since ΣP_3 and $X \vee X$ are the homotopy pushouts of $* \leftarrow P_3 \to *$ and $X \leftarrow * \to X$ respectively, we have a homotopy pushout diagram:

Diagram 4.3.

$$\begin{tikzcd}
\Sigma P_3 \arrow[r, "r_2"] \arrow[d, "q_3"] & X \arrow[d] \\
X \vee X \arrow[r, "g_3"] & M_3
\end{tikzcd}$$

where r_2 is the map induced by the top face of Diagram 4.1. By construction, the cube of Diagram 4.1 is Cartesian. Each terminal 2-dimensional face is co-Cartesian (here we need that X is a 1-connected co-H-space) and hence $(2n-3)$-Cartesian by Proposition 3.2 since j is $(n-1)$-connected. Since μ, i_1 and i_2 are $(n-1)$-connected the cube is $(3n-2)$-co-Cartesian by Proposition 3.3. In particular, the induced map $r_3 : M_3 \to X \vee X$ is $(3n-2)$-connected. Since $r_3 \circ g_3 \simeq \mathrm{id}$, the map g_3 is $(3n-3)$-connected. Hence r_2 is homology $(3n-3)$-connected because Diagram 4.3 is a homotopy pushout. Since ΣP_3 and X are simply connected r_2 is homotopy $(3n-3)$-connected. We summarize

Proposition 4.4. *Let X be an $(n-1)$-connected CW-complex with a co-H-space structure, $n \geqslant 2$. Then $\Sigma P_3 \to X$ is $(3n-3)$-connected. Hence there is a CW-complex K and a homotopy equivalence $\Sigma K \simeq X$.*

For the second part we refer the reader to the proof of Theorem 1.2 at the end of Section 6.

5. Cubical diagrams of co-A_k-spaces

In this section we construct Diagram 4.1 in the general case. The category $\mathcal{WG}$ of Example 2.1 only codifies the coherence of comultiplications. To incorporate inclusions such as $i_1, i_2 : X \to X \vee X$ or $j : * \to X$ we have to enlarge it to a category $\mathcal{W}^*\mathcal{G}$. We define ob: $\mathcal{W}^*\mathcal{G} = \operatorname{ob} \mathcal{WG}$ and

$$\mathcal{W}^*\mathcal{G}(n,k) = \coprod_l \mathcal{WG}(l,k) \times \mathrm{Inj}(l,n)$$

where $\mathrm{Inj}(l,n)$ is the set of ordered injections $\underline{l} \to \underline{n}$. In particular, $\mathcal{W}^*\mathcal{G}(n,0)$ consists of a single element $(\mathrm{id}_0, \emptyset \to \underline{n})$.

Composition of $(f,\alpha) \in \mathcal{WG}(l,k) \times \mathrm{Inj}(l,n)$ with

$$(g_1 \oplus \cdots \oplus g_n, \beta) \in \mathcal{WG}(m,n) \times \mathrm{Inj}(m,p)$$

is defined by

$$(f,\alpha)\circ(g_1\oplus\cdots\oplus g_n,\beta)=(f\circ(g_{\alpha(1)}\oplus\cdots\oplus g_{\alpha(l)}),\beta\circ\alpha(m_1,\ldots,m_n)).$$

Here $g_i\in\mathcal{WG}(m_i,1)$, $m=m_1+\cdots+m_n$, and

$$\alpha(m_1,\ldots,m_n):\underline{m_{\alpha(1)}+\cdots+m_{\alpha(l)}}\to\underline{m}$$

is the injection sending $m_{\alpha(1)}+\cdots+m_{\alpha(i-1)}+j$ to $m_1+m_2+\cdots+m_{\alpha(i)-1}+j$ for $1\leqslant j\leqslant m_{\alpha(i)}$, $1\leqslant i\leqslant l$.

The bifunctor $\oplus$ extends to $\mathcal{W}^*\mathcal{G}$ by

$$(f,\alpha)\oplus(g,\beta)=(f\oplus g,\alpha\sqcup\beta)$$

where $\alpha\sqcup\beta$ is the ordered disjoint union.

Let $\mathcal{W}_k^*\mathcal{G}$ denote the full subcategory of $\mathcal{W}^*\mathcal{G}$ consisting of all objects $j\leqslant k$. A co-A_k-space $X:(\mathcal{W}_k\mathcal{G})^{\mathrm{op}}\to\mathcal{T}op^*$ extends to a continuous functor

$$X^*:(\mathcal{W}_k^*\mathcal{G})^{\mathrm{op}}\to\mathcal{T}op^*$$

mapping $\oplus$ to $\vee$ by

$$X^*(f,\alpha):X^{\vee k}\xrightarrow{X(f)}X^{\vee l}\xrightarrow{\alpha_*}X^{\vee n}$$

for $(f,\alpha)\in\mathcal{WG}(l,k)\times\mathrm{Inj}(l,n)$, where α_* maps the ith summand identically onto the $\alpha(i)$th one (checking of functoriality is left to the reader).

In an analogous way we can adjoin the injections to $\mathcal{G}$ to obtain a category $\mathcal{G}^*$. We have the following fact:

Proposition 5.1. *The augmentation functor $\varepsilon:\mathcal{WG}\to\mathcal{G}$ and its section $\eta:\mathcal{G}\to\mathcal{WG}$ extend to a functor $\varepsilon^*:\mathcal{W}^*\mathcal{G}\to\mathcal{G}^*$ and a nonfunctorial section $\eta^*:\mathcal{G}^*\to\mathcal{W}^*\mathcal{G}$. Again ε^* is a homotopy equivalence on morphism spaces with homotopy inverse η^*.*

For a co-H-space we started with the commutative $\mathcal{T}_3$-diagram 4.1. For a co-A_{k-1}-space with $k-1\geqslant 3$ associating homotopies enter the picture, and we only can hope for a coherently homotopy commutative $\mathcal{T}_k$-diagram to start with. To describe this we need a variant of Example 2.1:

Let $\mathcal{D}$ be a small category. Homotopy coherent $\mathcal{D}$-diagrams are codified by the topological category $\mathcal{WD}$ [2, Chapter VII] with $\mathrm{ob}\,\mathcal{WD}=\mathrm{ob}\,\mathcal{D}$ and

$$\mathcal{WD}(A,B)=\Bigl(\coprod_n\mathcal{D}_{n+1}(A,B)\times I^n\Bigr)\Big/\sim$$

where $\mathcal{D}_{n+1}(A,B)$ is the set of $(n+1)$-tuples $(f_n,\ldots,f_0)$ of morphisms in $\mathcal{D}$ such that $f_n\circ\cdots\circ f_0:A\to B$ is defined. The relations are

$$(f_n,t_n,\ldots,f_1,t_1,f_0)=\begin{cases}(f_{n-1},t_{n-1},\ldots,f_0) & \text{if } f_n=\mathrm{id},\\ (f_n,\ldots,f_{i+1},\max(t_{i+1},t_i),f_{i-1},\ldots,f_0) & \text{if } f_i=\mathrm{id},\\ (f_n,t_n,\ldots,t_2,f_1) & \text{if } f_0=\mathrm{id},\\ (f_n,\ldots,f_{i+1}\circ f_i,t_i,\ldots,f_0) & \text{if } t_{i+1}=0.\end{cases}$$

Composition is defined by

$$(f_n, t_n, \dots, f_0) \circ (g_p, u_p, \dots, g_0) = (f_n, t_n, \dots, f_0, 1, g_p, u_p, \dots, g_0)$$

$\mathcal{WD}$ can be viewed as obtained from $\mathcal{D}$ by taking the free category over the diagram $\mathcal{D}$ and putting back the relations up to coherent homotopies.

Lemma 5.2. *The augmentation* $\varepsilon : \mathcal{WD} \to \mathcal{D}$, $(f_n, t_n, \dots, f_0) \mapsto f_n \circ \cdots \circ f_0$ *is a homotopy equivalence of morphism spaces with the nonfunctorial section* $\eta : \mathcal{D} \to \mathcal{WD}$, $f \mapsto (f)$ *as homotopy inverse* [2, *Proposition* 3.15].

Definition 5.3 [2]. A *homotopy coherent $\mathcal{D}$-diagram* is a continuous functor $D : \mathcal{WD} \to \mathcal{T}op^*$.

Example 5.4. A homotopy $\mathcal{Q}_2$-diagram $X : \mathcal{Q}_2 \to \mathcal{T}op^*$ is given by

$$\begin{tikzcd}
X_\emptyset \arrow[r, "X(\eta(f))"] \arrow[d, "X(\eta(g))"'] \arrow[dr, "X(\eta(h))"] & X_{\{2\}} \arrow[d, "X(\eta(\overline{g}))"] \\
X_{\{1\}} \arrow[r, "X(\eta(\overline{f}))"] & X_{\{1,2\}}
\end{tikzcd}$$

and two based homotopies $F : X(\eta(h)) \simeq X(\eta(\overline{g})) \circ X(\eta(f))$ and $G : X(\eta(h)) \simeq X(\eta(\overline{f})) \circ X(\eta(g))$ which are part of the structure. Here $\eta : \mathcal{Q}_2 \to \mathcal{WQ}_2$ is the canonical section and the morphisms f, $\overline{f}$, g, $\overline{g}$, and h in $\mathcal{Q}_2$ are the inclusions of the indexing sets of the spaces at the vertices. Obviously, a homotopy commutative square with a specified homotopy induces a homotopy $\mathcal{Q}_2$-diagram and vice versa.

Before we can construct our homotopy $\mathcal{T}_k$-diagram we need another functor which will turn out to be also useful for the proof of Theorem 1.4.

Construction 5.5. Let $\triangle^{\mathrm{inj}}$ denote the category of ordered sets $[n] = \{0, 1, \dots, n\}$ and order preserving injections and $\triangle^{\mathrm{inj}}_k$ the full subcategory of objects $[j]$, $j \leqslant k$. Define a functor

$$\vartheta_k : \mathcal{T}_k \to \triangle^{\mathrm{inj}}_{k-1}$$

by $\vartheta_k(\{i_1, \dots, i_p\}) = [p-1]$. Identify $i_1 < \cdots < i_p$ in order with $0 < 1 < \cdots < p-1$. Then an inclusion $\alpha : \{i_1, \dots, i_p\} \subset \{j_1, \dots, j_q\}$ defines a unique order preserving map $\vartheta_k(\alpha) : [p-1] \to [q-1]$. (We always list the elements of a subset of $\underline{n}$ in increasing order.)

Construction 5.6. $\varphi_k : \triangle^{\mathrm{inj}}_k \to (\mathcal{G}^*_k)^{\mathrm{op}}$ is defined by sending $[j]$ to j and $\alpha : [p] \to [q]$ to

$$\left(\bigoplus_{i=1}^{p} \lambda_{\alpha(i)-\alpha(i-1)}, \sigma_\alpha \right),$$

where $\sigma_\alpha : \underline{\alpha(p) - \alpha(0)} \to \underline{q}$ is the ordered injection missing the elements $1, \ldots, \alpha(0)$, $\alpha(p)+1, \ldots, q$.

Construction 5.7. We now turn to the actual construction of a continuous functor

$$\psi_k : \mathcal{WT}_k \to \mathcal{T}op^*.$$

Let $X : (\mathcal{W}_{k-1}\mathcal{G})^{\mathrm{op}} \to \mathcal{T}op^*$ be a co-A_{k-1}-space. Then ψ_k is a composite

$$\psi_k : \mathcal{WT}_k \xrightarrow{\theta_k} (\mathcal{W}^*_{k-1}\mathcal{G})^{\mathrm{op}} \xrightarrow{X^*} \mathcal{T}op^*$$

and hence functorial in X. We need all terminal faces of ψ_k highly co-Cartesian. To achieve this we construct θ_k in such a way that all 2-dimensional faces of $X^* \circ \theta_k$ are co-Cartesian. It then follows that all higher dimensional faces of ψ_k are co-Cartesian too; a diagram of this kind is called *strongly co-Cartesian* in [6].

As explained in Example 5.4 it suffices to specify θ_k on all objects and on morphisms $\eta(f)$ for $f : \{i_1, \ldots, i_p\} \subset \{l_1, \ldots, l_{p+1}\}$ in $\mathcal{T}_k$ and to define homotopies $\eta(\overline{g}) \circ \eta(f) \simeq \eta(\overline{f}) \circ \eta(g)$ for $\overline{g} \circ f = \overline{f} \circ g : \{i_1, \ldots, i_p\} \subset \{j_1, \ldots, j_{p+2}\}$. To simplify notation we write f for $\eta(f)$ if there is no chance for confusion.

On objects and on morphisms f as above θ_k is given by $\eta^* \circ \varphi_{k-1} \circ \vartheta_k \circ \varepsilon$ where $\varepsilon : \mathcal{WT}_k \to \mathcal{T}_k$ is the canonical augmentation and $\eta^* : (\mathcal{G}^*_{k-1})^{\mathrm{op}} \to (\mathcal{W}^*_{k-1}\mathcal{G})^{\mathrm{op}}$ the canonical section. In detail:

$$\theta_k(\{i_1, \ldots, i_p\}) = p - 1.$$

If $p = 1$ we take $\theta_k(f) = (\mathrm{id}_0, \emptyset \to \underline{1})$, and if $p > 1$

$$\theta_k(f) = \begin{cases} \iota_1 \oplus \mathrm{id}_{p-2} & \text{if } l_1 \notin \operatorname{Image} f, \\ \mathrm{id}_{r-2} \oplus \lambda_2 \oplus \mathrm{id}_{p-r} & \text{if } l_r \notin \operatorname{Image} f,\ 1 < r < p+1, \\ \mathrm{id}_{p-2} \oplus \iota_2 & \text{if } l_{p+1} \notin \operatorname{Image} f, \end{cases}$$

where $\iota_r : \underline{1} \to \underline{2}$ denotes the injection missing the rth element. We now define the homotopies: Let $h : \{i_1, \ldots, i_p\} \subset \{j_1, \ldots, j_{p+2}\}$ be a morphism in $\mathcal{T}_k$ and $j_r < j_q$ the elements not in the image of h. Then h decomposes in $\mathcal{T}_k$ into

$$\begin{aligned} h &= \overline{g} \circ f : \{i_1, \ldots, i_p\} \subset \{j_1, \ldots, \hat{j}_q, \ldots, j_{p+2}\} \subset \{j_1, \ldots, j_{p+2}\} \\ &= \overline{f} \circ g : \{i_1, \ldots, i_p\} \subset \{j_1, \ldots, \hat{j}_r, \ldots, j_{p+2}\} \subset \{j_1, \ldots, j_{p+2}\} \end{aligned}$$

where $\hat{j}_i$ means that this element is deleted.

Case 5.7.1. If $1 < r < q < p+2$ we have a diagram

$$\begin{tikzcd} p-1 \arrow[r, "\mathrm{id}_{r-2}\oplus\lambda_2\oplus\mathrm{id}_{p-r}"] \arrow[d, "\mathrm{id}_{q-3}\oplus\lambda_2\oplus\mathrm{id}_{p+1-q}"'] & p \arrow[d, "\mathrm{id}_{q-2}\oplus\lambda_2\oplus\mathrm{id}_{p+1-q}"] \\ p \arrow[r, "\mathrm{id}_{r-2}\oplus\lambda_2\oplus\mathrm{id}_{p+1-r}"] & p+1 \end{tikzcd}$$

which commutes if $q > r+1$.

If $q = r+1$ this diagram reduces to the form

$$\mathrm{id}_{r-2} \oplus \begin{tikzcd} 1 \arrow[r, "\lambda_2"] \arrow[d, "\lambda_2"] & 2 \arrow[d, "\mathrm{id}\oplus\lambda_2"] \\ 2 \arrow[r, "\lambda_2\oplus\mathrm{id}"] & 3 \end{tikzcd} \oplus \mathrm{id}_{p-r}.$$

The square commutes up to homotopy by the associating homotopy

$$\eta^*(\mathrm{id} \oplus \lambda_2) \circ \eta^*(\lambda_2) \simeq \eta^*(\lambda_3) \simeq \eta^*(\lambda_2 \oplus \mathrm{id}) \circ \eta^*(\lambda_2).$$

Case 5.7.2. If $r = 1, q = 2$ we have a commutative square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d, "\lambda_2"] \\ 1 \arrow[r, "\iota_1"] & 2 \end{tikzcd} \oplus \mathrm{id}_p$$

and similarly for $r = p+1$, $q = p+2$.

Case 5.7.3. If $r = 1$ and $q = p+2$ we have commutative squares

$$\begin{tikzcd} p-1 \arrow[r, "\iota_1\oplus\mathrm{id}_{p-2}"] \arrow[d, "\mathrm{id}_{p-2}\oplus\iota_2"] & p \arrow[d, "\mathrm{id}_{p-1}\oplus\iota_2"] \\ p \arrow[r, "\iota_1\oplus\mathrm{id}_{p-1}"] & p+1 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d, "\iota_2"] \\ 1 \arrow[r, "\iota_1"] & 2 \end{tikzcd}$$

for $p > 1$, respectively $p = 1$.

The diagrams of Cases 5.7.2 and 5.7.3 also take care of the case $p = 1$.

Case 5.7.4. If $r = 1$ and $2 < q < p+2$ we have a commutative square

$$\begin{tikzcd} p-1 \arrow[r, "\iota_1\oplus\mathrm{id}_{p-2}"] \arrow[d, "\mathrm{id}_{q-3}\oplus\lambda_2\oplus\mathrm{id}_{p+1-q}"'] & p \arrow[d, "\mathrm{id}_{q-2}\oplus\lambda_2\oplus\mathrm{id}_{p+1-q}"] \\ p \arrow[r, "\iota_1\oplus\mathrm{id}_{p-1}"] & p+1 \end{tikzcd}$$

and similarly for $1 < r < p+1$ and $q = p+2$.

Let $\mathcal{V} \subset \mathcal{W}\mathcal{T}_k$ be the subcategory generated by all $\mathcal{W}\mathcal{Q}_2 \subset \mathcal{W}\mathcal{T}_k$ where $\mathcal{Q}_2$ runs through all 2-dimensional faces of $\mathcal{T}_k$. Then our construction defines θ_k on $\mathcal{V}$. We now apply the lifting theorem [2, 3.17] to

$$\begin{tikzcd}
\mathcal{V} \arrow[d, phantom, "\cap"] \arrow[drr, "\theta_k|\mathcal{V}"] & & \\
\mathcal{W}\mathcal{T}_k \arrow[d, "\varepsilon"] & & (\mathcal{W}^*_{k-1}\mathcal{G})^{\mathrm{op}} \arrow[d, "\varepsilon^*"] \\
\mathcal{T}_k \arrow[rr, "\varphi_{k-1}\circ\vartheta_k"] & & (\mathcal{G}^*_{k-1})^{\mathrm{op}}
\end{tikzcd}$$

to obtain an extension of $\theta_k|\mathcal{V}$ to $\mathcal{W}\mathcal{T}_k$.

Lemma 5.8. *Let X be a 1-connected co-A_{k-1}-space, $k \geqslant 3$. Then for each $(k-1)$-dimensional face $\mathcal{S}$ of $\mathcal{T}_k$ the homotopy coherent $\mathcal{S}$-diagram*

$$\psi_k|\mathcal{W}\mathcal{S} : \mathcal{W}\mathcal{S} \to \mathcal{T}op^*$$

is strongly (homotopy) co-Cartesian in the sense of [6, 2.1].

Proof. We have to show that the specified faces of the functor θ_k are homotopy pushouts after composition with X^*. This holds for the first diagram of Case 5.7.1, and for Case 5.7.4 if $r = 1$ and $3 \leqslant q$ or $r \leqslant p$ and $q = p + 2$. For given maps $f : A \to B$ and $g : C \to D$ in $\mathcal{T}op^*$, the square

$$\begin{tikzcd}
A \vee C \arrow[r, "f\vee\mathrm{id}"] \arrow[d, "\mathrm{id}\vee g"] & B \vee C \arrow[d, "\mathrm{id}\vee g"] \\
A \vee D \arrow[r, "f\vee\mathrm{id}"] & B \vee D
\end{tikzcd}$$

is always a homotopy pushout. Case 5.7.3 holds by direct investigation. For the second diagram of Case 5.7.1 we use the fact that a 1-connected homotopy associative co-H-space which is a CW-complex has a 2-sided inverse [5, Proposition 3.6]. For Case 5.7.2 we appeal to [19, Lemma 3.3]. □

6. Proof of Theorem 1.2

Recall from Diagrams 4.2 and 4.3 that the homotopy limit P_3 of the $\mathcal{W}\mathcal{T}_3$-diagram we started with can be obtained as an iterated homotopy pullback and that the homotopy colimit M_3 of the induced $\mathcal{W}\mathcal{J}_3$-diagram can be obtained as an iterated homotopy pushout.

We mimic this in the general case. So let $\psi_l : \mathcal{W}\mathcal{T}_l \to \mathcal{T}op^*$, $2 \leqslant l \leqslant k$ be the cube diagrams determined by the co-A_{l-1}-structures of the co-A_{k-1}-space X. Let $\mathcal{S}_{k-1} \subset \mathcal{T}_k$ be the face defined by all subsets of $\underline{k}$ containing the element 1 (it corresponds to the bottom face of Diagram 4.1), and $\mathcal{TS}_{k-1}$ obtained from it by deleting the initial vertex. $\mathcal{TS}_{k-1}$ is the subdiagram of all proper terminal faces of $\mathcal{S}_{k-1}$. Let P_l denote the homotopy limit (in the sense of [16]) of the coherently homotopy commutative $\mathcal{T}_l$-diagram ψ_l and R_{k-1} the one of the subdiagram $\mathcal{WTS}_{k-1} \to \mathcal{T}op^*$. Observe that P_l and R_{k-1} are still well-pointed by [16, Proposition 6.9]. We can obtain P_k as the homotopy limit of the induced diagram

$$* = \psi_k\{1\} \to R_{k-1} \leftarrow P_{k-1}$$

(compare [18, Proposition 5.4]). In particular, we have a sequence of fibrations

$$P_k \xrightarrow{p_k} P_{k-1} \xrightarrow{p_{k-1}} \cdots \xrightarrow{p_3} P_2 = \Omega X$$

with fiber$(p_k) = \Omega R_{k-1}$. Since $\psi_k|\mathcal{WS}_{k-1}$ is strongly co-Cartesian and each map $\psi_k(\underline{k}\backslash\{i\} \to \underline{k}) : X^{\vee(k-2)} \to X^{\vee(k-1)}$ is $(n-1)$-connected, the induced map $* \to R_{k-1}$ is $(1-(k-1)+(k-1)(n-1))$-connected by Proposition 3.2, and so is p_k. Since $n \geqslant 2$ and $P_2 = \Omega X$ each P_k, $k \geqslant 2$, is connected.

P_k together with the universal transformation $P_k \to \psi_k$ extends ψ_k to a $\mathcal{WQ}_k$-diagram $\Theta_k : \mathcal{WQ}_k \to \mathcal{T}op^*$, i.e., to a homotopy coherent k-dimensional cube diagram. Let $\widetilde{\mathcal{T}} \subset \mathcal{Q}_k$ be an l-dimensional terminal face, $l < k$. By Lemma 5.8 it is strongly co-Cartesian, and hence by Proposition 3.2 $(1 + l(n-2))$-Cartesian. Since Θ_k is infinitely Cartesian it is $(k-1+2+k(n-2))$-co-Cartesian by Proposition 3.3.

Recall that $\mathcal{J}_l \subset \mathcal{Q}_k$ is the subcategory of all subsets of $\underline{l} \subset \underline{k}$, $l \leqslant k$, except for $\underline{l}$ itself. Let $\mathcal{I}_{l-1} \subset \mathcal{J}_l$ denote the subcategory of all subsets containing 1 and $\mathcal{JI}_{l-1}$ the full subcategory of $\mathcal{I}_{l-1}$ containing all but the terminal element, and let M_l denote the homotopy colimit of $\Theta_k|\mathcal{WJ}_l$.

By Lemma 5.8 the homotopy colimit of $\Theta_k|\mathcal{WJI}_{l-1}$ is $X^{\vee(l-1)}$ and the induced map to the terminal vertex of $\mathcal{I}_{l-1}$ is a homotopy equivalence. By [18, Proposition 5.4] we have homotopy pushout diagrams:

Diagram 6.1.

$$\begin{tikzcd}
M_{l-1} \arrow[r, "r_{l-1}"] \arrow[d, "q_l"'] & X^{\vee(l-2)} \arrow[d] \\
X^{\vee(l-1)} \arrow[r, "g_l"] & M_l
\end{tikzcd}$$

for $3 \leqslant l \leqslant k$, where $r_l : M_l \to X^{\vee(l-1)}$ is the induced map into the terminal vertex of $\Theta_k|\mathcal{WQ}_l$. Then, as noted above, $r_l \circ g_l \simeq \mathrm{id}$. Since Θ_k is $(k(n-2)+k+1)$-co-Cartesian, r_k is $(k(n-2)+k+1)$-connected, hence g_k is $(k(n-2)+k)$-connected. Since Diagram 6.1 is a homotopy pushout, r_{k-1} is homology $(k(n-2)+k)$-connected. By downwards induction we obtain

$$r_2 : \Sigma P_k = M_2 \to X$$

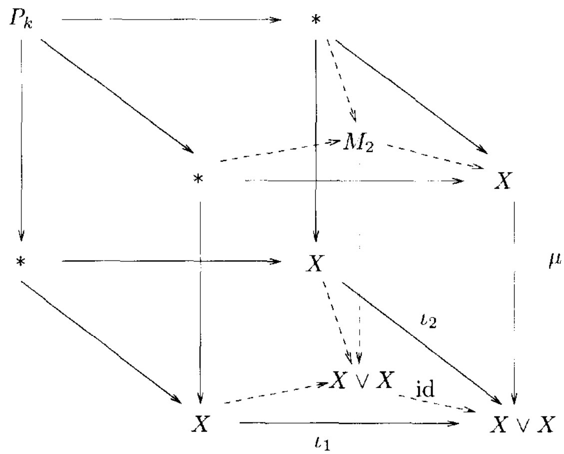
is homology $(k(n-2)+3)$-connected. Since ΣP_k and X are both 1-connected, this implies that r_2 is $(k(n-2)+3)$-connected. We have proved the connectivity part of

Proposition 6.2. *Let X be an $(n-1)$-connected co-A_{k-1}-space, $n \geqslant 2$, $k \geqslant 3$. Let P_k be the homotopy limit of its associated coherently homotopy commutative $\mathcal{T}_k$-diagram. Then the induced map*

$$r_2 : \Sigma P_k \to X$$

is a $(k \cdot (n-2)+3)$-connected co-A_{k-1}-homomorphism.

It remains to show that $r_2 : \Sigma P_k = M_2 \to X$ is a co-A_{k-1}-homomorphism. The first of Diagrams 6.1 is obtained from (see also Diagram 4.3)



with $\mu = X(\eta(\lambda_2))$. The induced map $q_3 : M_2 = \Sigma P_k \to X \vee X$ factors through the standard pinch map and a wedge of two maps so that we arrive at a homotopy commutative diagram

$$\begin{tikzcd}
\Sigma P_k \arrow[r, "r_2"] \arrow[d, "\text{pinch}"] & X \arrow[dd] \arrow[ddr, "\mu"] & \\
\Sigma P_k \vee \Sigma P_k \arrow[d, "f \vee g"] & & \\
X \vee X \arrow[r, "g_3"] & M_3 \arrow[r, "r_3"] & X \vee X
\end{tikzcd}$$

Since $r_3 \circ g_3 \simeq \mathrm{id}$ and the pinch map and μ have counits we deduce $f \simeq r_2 \simeq g$. Hence r_2 is a homomorphism of co-H-spaces up to homotopy (to obtain higher coherence one has to include the higher dimensional cubes into the argument; we leave this to the reader).

Proof of Theorem 1.2. Let K be the CW-approximation of P_k, and suppose that X is a CW-complex with $\dim X \leqslant k(n-2)+3$. Since $H_{k(n-2)+3}X$ is free abelian and r_2 induces a map $q : \Sigma K \to X$ which is a homology isomorphism in dimensions less than $k(n-2)+3$ and a homology epimorphism in dimension $k(n-2)+3$, there is a $(k(n-2)+2)$-dimensional CW-complex Y having the same $(k(n-2)+1)$-skeleton as K and a map $f : Y \to K$ such that $q \circ \Sigma f : \Sigma Y \to X$ is a homology isomorphism (cf. [1, Theorem 2.1]). Moreover, $q \circ \Sigma f$ is a co-A_{k-1}-map. Since ΣY and X are 1-connected it is a homotopy equivalence.

7. Proof of Theorem 1.4

Let $X : \triangle \to \mathcal{T}op^*$ be a based cosimplicial space such that X_0 is contractible, X_1 is 2-connected, and

$$(\pi_1, \ldots, \pi_n) : \bigvee_{k=1}^{n} X_1 \to X_n$$

is a homotopy equivalence. Define

$$\mu : X \xrightarrow{d^1} X_2 \longrightarrow X \vee X$$

where the second map is a homotopy inverse of $(\pi_1, \pi_2) = (d^2, d^0)$. (By abuse of notation we write X for X_1.) Then (X, μ) is a co-H-space because

$$\begin{tikzcd}
X_2 \arrow[d, "s^1"'] & X \arrow[l, "d^1"'] \arrow[r, "d^1"] \arrow[d, "\mu"] & X_2 \arrow[d, "s^0"] \\
X & X \vee X \arrow[l, "\mathrm{proj}_1"] \arrow[r, "\mathrm{proj}_2"'] \arrow[ul, "(d^2, d^0)"'] \arrow[ur, "(d^2, d^0)"] & X
\end{tikzcd}$$

commutes up to homotopy since X_0 is contractible. In a similar way one verifies that (X, μ) is homotopy associative.

Lemma 7.1. *Each l-dimensional face, $l \geqslant 2$, of $X \circ \vartheta_k : \mathcal{T}_k \to \triangle^{\mathrm{inj}}_{k-1} \to \mathcal{T}op^*$ is strongly homotopy co-Cartesian.*

Proof. We evaluate $X \circ \vartheta_k$ on 2-dimensional faces. Let $\alpha : \{i_1, \ldots, i_p\} \subset \{j_1, \ldots, j_{p+2}\}$ be an inclusion. If α misses $j_r < j_q$ the associated square is

Diagram 7.2.

$$\begin{tikzcd}
X_{p-1} \arrow[r, "d^{r-1}"] \arrow[d, "d^{q-2}"'] & X_p \arrow[d, "d^{q-1}"] \\
X_p \arrow[r, "d^{r-1}"] & X_{p+1}
\end{tikzcd}$$

Each such square is homotopy equivalent to a square considered in the proof of Lemma 5.8 and hence a homotopy pushout: for $0 < i < p+1$ consider

Diagram 7.3.

$$\begin{tikzcd}
 & X^{\vee(i-1)} \vee X \vee X^{\vee(p-i)} \arrow[r, "\simeq"] \arrow[dl, "\mathrm{id}\vee\mu\vee\mathrm{id}"'] \arrow[d, "\mathrm{id}\vee d^1\vee\mathrm{id}"] & X_p \arrow[d, "d^i"] \\
X^{\vee(i-1)} \vee (X\vee X) \vee X^{\vee(p-i)} \arrow[r, "\mathrm{id}\vee(d^2,d^0)\vee\mathrm{id}"] & X^{\vee(i-1)} \vee X_2 \vee X^{\vee(p-i)} \arrow[r, "\simeq"] & X_{p+1}
\end{tikzcd}$$

The map $\mathrm{id} \vee (d^2, d^0) \vee \mathrm{id} = \mathrm{id} \vee (\pi_1, \pi_2) \vee \mathrm{id}$ is an equivalence by assumption on X, and so are the other horizontal equivalences. The square commutes while the triangle commutes up to homotopy. Hence, if $1 < r < q < p+2$, Diagram 7.2 is equivalent to

$$\begin{tikzcd}
X^{\vee(p-1)} \arrow[r, "\mathrm{id}_{r-2}\vee\mu\vee\mathrm{id}_{p-r}"] \arrow[d, "\mathrm{id}_{q-3}\vee\mu\vee\mathrm{id}_{p+1-q}"'] & X^{\vee p} \arrow[d, "\mathrm{id}_{q-2}\vee\mu\vee\mathrm{id}_{p+1-q}"] \\
X^{\vee p} \arrow[r, "\mathrm{id}_{r-2}\vee\mu\vee\mathrm{id}_{p+1-r}"] & X^{\vee(p+1)}
\end{tikzcd}$$

which corresponds to Case 5.7.1, and hence is a homotopy pushout, because (X, μ) is a homotopy associative co-H-space. For $d^0, d^{p+1} : X_p \to X_{p+1}$ the squares corresponding to Diagram 7.3 arise from

$$\begin{tikzcd}
X \vee X_p \arrow[d, "(\pi_1,d^0)"', "\simeq"] & X_p \arrow[l, "\mathrm{incl.}"'] \arrow[r, "\mathrm{incl.}"] \arrow[d, equal] & X_p \vee X \arrow[d, "\simeq"', "(d^{p+1},\pi_{p+1})"] \\
X_{p+1} & X_p \arrow[l, "d^0"'] \arrow[r, "d^{p+1}"] & X_{p+1}
\end{tikzcd}$$

As before, the remaining cases of Diagram 7.2 give diagrams of Cases 5.7.2–5.7.4 and hence are homotopy pushouts. □

Proof of Corollary 1.3 and Theorem 1.4. Let P_k denote the homotopy limit of $X \circ \vartheta_k$. As shown in the proof of Theorem 1.2 we obtain a sequence of fibrations

$$\cdots \longrightarrow P_4 \xrightarrow{p_4} P_3 \xrightarrow{p_3} P_2 \simeq \Omega X$$

such that $p_k : P_k \to P_{k-1}$ is k-connected because X is 2-connected, and a $(k+3)$-connected map $r_k : \Sigma P_k \to X$. Let P be the homotopy limit of the P_k. Since

$$\lim{}^1 \pi_i(P_k) = 0$$

we have

$$\pi_k(P) \cong \lim \pi_k(P_i) \cong \pi_k(P_k).$$

In particular, the natural projection $q_k : P \to P_k$ is k-connected. The maps $r_k : \Sigma P_k \to X$ are compatible with the p_k up to homotopy and induce a map

$$\Sigma P \to X.$$

Since $\Sigma P \to \Sigma P_k \to X$ is $(k+1)$-connected, this map is a weak equivalence.

Acknowledgment

We thank the Deutsche Forschungsgemeinschaft for support, M. Golasinski for bringing Saito's paper to our attention, and the referee for helpful comments concerning the presentation of the paper.

References

[1] I. Bernstein and P.J. Hilton, On suspension and comultiplications, Topology 2 (1963) 73–82.
[2] J.M. Boardman and R.M. Vogt, Homotopy Invariant Structures on Topological Spaces, Lecture Notes in Math. 347 (1973).
[3] A.K. Bousfield, On the homology spectral sequence of a cosimplicial space, Amer. J. Math. 109 (1987) 361–394.
[4] G. Ellis and R. Steiner, Higher-dimensional crossed modules and the homotopy groups of $(n+1)$-ads, J. Pure Appl. Algebra 46 (1987) 117–136.
[5] T. Ganea, Cogroups and suspensions, Invent. Math. 9 (1970) 185–197.
[6] T. Goodwillie, Calculus II: Analytic functors, K-Theory 5 (1992) 295–332.
[7] M.J. Hopkins, Formulations of cocategory and the iterated suspension, Astérisque 113–114 (1984) 212–226.
[8] J.R. Klein, Embedding theory in the Poincaré category, in preparation.
[9] J.R. Klein, R. Schwänzl and R.M. Vogt, Parameterized suspension theory, in preparation.
[10] J.P. May, The Geometry of Iterated Loop Spaces, Lecture Notes in Math. 271 (1972).
[11] S. Saito, On higher coassociativity, Hiroshima Math. J. 6 (1976) 589–617.
[12] G. Segal, Categories and cohomology theories, Topology 13 (1974) 293–312.
[13] J.D. Stasheff, Homotopy associativity of H-spaces I, Trans. Amer. Math. Soc. 108 (1963) 293–312.
[14] J.D. Stasheff, H-spaces from the Homotopy Point of View, Lecture Notes in Math. 161 (1970).
[15] R.M. Vogt, Convenient categories of topological spaces for homotopy theory, Arch. Math. 22 (1971) 545–555.
[16] R.M. Vogt, Homotopy limits and colimits, Math. Z. 134 (1973) 11–52.
[17] R.M. Vogt, A remark on A_n-spaces and loop spaces, J. London Math. Soc. (2) 14 (1976) 321–325.
[18] R.M. Vogt, Commuting homotopy limits, Math. Z. 153 (1977) 59–82.
[19] M. Walker, Homotopy pull-backs and applications to duality, Canad. J. Math. 29 (1977) 45–64.