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On maps into a co-H-space

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ABSTRACT. We prove that the domain of a map $X \to Y$ to a co-H-space inherits a co-H-structure provided some dimensionality and connectivity properties hold. Then we deduce that a space X admits a co-H-structure if and only if on all its skeletons there is such a structure as well. Moreover, if X is 1-connected then a co-H-structure on X is equivalent to such a structure on all its homology decomposition stages.

Introduction.

Recall [5, Chapter IX], where it was shown that a map $X \to Y$ of based spaces extends an H-space structure on X to one on Y provided some connectivity properties hold. In particular, the result implies that an H-structure on X is inherited on all its Postnikov stages.

This paper examines the dual problem. However, the arguments used for H-spaces do not seem to dualize. Therefore, we make use of the one-to-one correspondence (see [1] or [2, p. 209–212]) between homotopy classes of co-H-structures on a space X and those of coretractions $X \to \Sigma \Omega X$. A result due to Hilton (see [2, p. 185]) says that if a space X is dominated by a co-H-space Y then X also admits a co-H-structure. We prove (Theorem 1) that the domain of a map $X \to Y$ to a co-H-space inherits a co-H-structure provided some dimensionality and connectivity properties hold. Then we deduce (Corollary 1) that a space X admits a co-H-structure if and only if on all its skeletons there is such a structure as well. Moreover, if X is 1-connected then Corollary 3 states that a co-H-structure on X is equivalent to such a structure on all its homology decomposition stages and a dualization of Corollary 5.6 in [5, page 443] is obtained provided X is 2-connected.

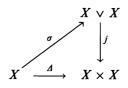
1. Preliminaries.

We consider based spaces of the based homotopy type of a based CWcomplex; the basepoints are assumed to be non-degenerate. All maps and

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homotopies are supposed to preserve basepoints and all spaces denoted by the symbols X, Y and their derivates $(X_n, Y_n \text{ etc.})$ are implicitly assumed to be based CW-complexes. It follows from [4] the constructions we will perform do not lead outside the class of spaces considered. For a space X let $\Delta: X \to X \times X$ be the diagonal map and $j: X \vee X \to X \times X$ the inclusion map of the wedge into the product. A *co-H-structure* on X is a map $\sigma: X \to X \vee X$ such that the diagram



commutes up to homotopy. A co-H-space is a pair (X, σ) , where X is a space and σ a co-H-structure on X. An *inversion* for σ is a map $\eta : X \to X$ such that the composites

$$X \xrightarrow{\sigma} X \lor X \xrightarrow{\operatorname{id}_X \lor \eta} X \lor X \xrightarrow{\nabla} X \quad \text{and} \quad X \xrightarrow{\sigma} X \lor X \xrightarrow{\eta \lor \operatorname{id}_X} X \lor X \xrightarrow{\nabla} X$$

are both nullhomotopic maps; ∇ is the folding map. We point out that by [3] any co-H-structure on a 1-connected space admits an inversion.

With any space X we may associate $\Sigma\Omega X$, the (reduced) suspension of the loop space of X; the natural projection $p_X : \Sigma\Omega X \to X$ is given by $p_X([t, \omega]) = \omega(t)$ for $[t, \omega] \in \Sigma\Omega X$. A coretraction is a map $\phi : X \to \Sigma\Omega X$ such that $p_X \phi \simeq id_X$, the identity map of X. For a space X the suspension co-Hstructure $s_X : \Sigma X \to \Sigma X \lor \Sigma X$ is the pinch map and the coretraction $\Sigma e : \Sigma X \to \Sigma\Omega\Sigma X$ is given by e(x)(t) = [t, x] for $x \in X$ and $t \in [0, 1]$. As it was pointed out in [1] and [2, p. 209-212] the map $(p_X \lor p_X)s_{\Omega X} :$ $\Sigma\Omega X \to X \lor X$ induces a bijection between homotopy classes of coretractions and those of comultiplications on X.

Let $f: X \to Y$ be any map and consider the diagrams

in which ϕ , ψ are coretractions and σ , τ the corresponding comultiplications. Taking into account [1] the second square homotopy commutes if and only if the first does; we way then that f is a map of co-H-spaces (X, σ) and (Y, τ) . We now show how maps from some spaces into a co-H-space reflect co-Hstructures on their domains. For a map $f: X \to Y$ we write conn f = n, if the induced map of homotopy groups $\pi_k(f): \pi_k(X) \to \pi_k(Y)$ is an isomorphism for k < n and an epimorphism for k = n. In particular, for the map $f: X \to *$ to the single point space we put conn X = conn f. Write $X \triangleright Y$ and $X \land Y$ for the flat and smash product of spaces X and Y, respectively; now there is a homotopy equivalence $X \triangleright Y \simeq \Sigma \Omega X \land \Omega Y$ (see e.g., [2, page 216]).

LEMMA 1. For a map $f: X \to Y$, $\operatorname{conn}(f \wedge f) = \operatorname{conn} f + \min\{\operatorname{conn} X, \operatorname{conn} Y\} + 1$ and $\operatorname{conn}(f \triangleright f) = \operatorname{conn} f + \min\{\operatorname{conn} X, \operatorname{conn} Y\}.$

PROOF. Observe that $f \wedge f = (f \wedge id_X) \circ (id_Y \wedge f)$ and $f \flat f = (f \flat id_X) \circ (id_Y \flat f)$. From the above $f \flat id_X \simeq \Omega f \wedge id_{\Sigma\Omega X}$ and $id_Y \flat f \simeq id_{\Sigma\Omega Y} \wedge \Omega f$. Hence $\operatorname{conn}(f \wedge id_X) = \operatorname{conn} f + \operatorname{conn} X + 1$, $\operatorname{conn}(id_Y \wedge f) = \operatorname{conn} f + \operatorname{conn} Y + 1$ and $\operatorname{conn}(f \flat id_X) = \operatorname{conn} X + \operatorname{conn} f$, $\operatorname{conn} id_Y \flat f = \operatorname{conn} Y + \operatorname{conn} f$. The result follows.

This lemma will be also useful in the next section.

LEMMA 2. Let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} \cdots$ be a countable direct system of co-H-spaces and co-H-maps. Then its homotopy colimit holim X_n admits a co-H-structure such that the canonical imbeddings $X_n \xrightarrow{f}$ holim X_n are co-H-maps for all $n \ge 0$.

PROOF. By the "small object" argument we get that the canonical map

holim
$$\Sigma \Omega X_n \longrightarrow \Sigma$$
 holim $\Omega X_n = \Sigma \Omega$ holim X_n

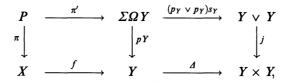
is a homotopy equivalence. Then the coretractions $\phi_n : X_n \to \Sigma \Omega X_n$ corresponding to co-H-structures on X_n for $n \ge 0$ yield the required coretraction $\phi = \operatorname{holim} \phi_n : \operatorname{holim} X_n \to \Sigma \Omega \operatorname{holim} X_n$.

2. Main results.

The behaviour of co-H-structures with respect to maps is taken into account in this section.

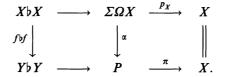
THEOREM 1. Let $f: X \to Y$ be a map with Y a co-H-space. If dim $X \leq \operatorname{conn} f + \min\{\operatorname{conn} X, \operatorname{conn} Y\}$ then there exists a (which is unique up to homotopy if strict inequality holds) co-H-structure on X such that $f: X \to Y$ is a co-H-map.

PROOF. Consider the commutative (up to homotopy) diagram



where the first square is the homotopy pullback and the second one is also a homotopy pullback by [1]. Therefore, the homotopy fibres of the maps $p_Y: \Sigma\Omega Y \to Y$ and $\pi: P \to X$ have the homotopy type of the space $Y \flat Y$. Let now $\phi: Y \to \Sigma\Omega Y$ be the coretraction corresponding to the co-Hstructure on Y. Then the pairs of maps $\Sigma\Omega f: \Sigma\Omega X \to \Sigma\Omega Y, p_X: \Sigma\Omega X \to X$ and $\phi f: X \to \Sigma\Omega Y$, id_X yield maps $\alpha: \Sigma\Omega X \to P$ and $\gamma: X \to P$, respectively with the appropriate composite property.

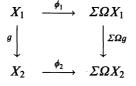
Analyze now the map of fibrations



From the 5-lemma we get that $\operatorname{conn} \alpha = \operatorname{conn} f \flat f$. But by Lemma 1 $\operatorname{conn} f \flat f = \operatorname{conn} f + \min \{\operatorname{conn} X, \operatorname{conn} Y\}$ and the obstruction theory yields a (unique up to homotopy if there is strict inequality) map $\psi : X \to \Sigma \Omega X$ such that $\alpha \psi \simeq \gamma$. Then $\pi \alpha \psi \simeq p_X \psi \simeq \pi \gamma \simeq \operatorname{id}_X$ and the map ψ determines a co-H-structure on X. Moreover, $(\Sigma \Omega f)\psi \simeq \pi' \alpha \psi \simeq \pi' \gamma \simeq \phi f$ and $f : X \to Y$ is a co-H-map.

LEMMA 3. Let $f_1: X_1 \to Y$, $f_2: X_2 \to Y$ and $g: X_1 \to X_2$ be maps with $f_{2g} \simeq f_1$ and Y a co-H-space. If $\dim X_1 \le \min\{\operatorname{conn} f_2 - 1, \operatorname{conn} f_1 + \min\{\operatorname{conn} X_1, \operatorname{conn} Y\}\}$ and $\dim X_2 \le \operatorname{conn} f_2 + \min\{\operatorname{conn} X_2, \operatorname{conn} Y\}$ then a co-H-structure on Y yields co-H-structures on X_1 and X_2 such that $g: X_1 \to X_2$ is a co-H-map.

PROOF. Let $\phi: Y \to \Sigma \Omega Y$ be the coretraction determined by the co-Hstructure on Y and $\phi_1: X_1 \to \Sigma \Omega X_1$, $\phi_2: X_2 \to \Sigma \Omega X_2$ the coretractions determined by the co-H-structures on X_1 and X_2 described in Theorem 1. To show the commutativity (up to homotopy) of the diagram



observe that $(\Sigma \Omega f_2)\phi_2 g \simeq \phi f_2 g \simeq \phi f_1$ and $(\Sigma \Omega f_2)(\Sigma \Omega g)\phi_1 \simeq (\Sigma \Omega f_2 g)\phi_1 \simeq (\Sigma \Omega f_1)\phi_1 \simeq \phi f_1$. But conn $\Sigma \Omega f_2 = \operatorname{conn} f_2$ and dim $X_1 \leq \operatorname{conn} f_2 - 1$, hence obstruction theory yields a homotopy $(\Sigma \Omega g)\phi_1 \simeq \phi_2 g$.

Our final results can be stated now. For a space X let $X^{(n)}$ be its *n*-skeleton, $n \ge 0$. Then Theorem 1, Lemmas 2 and 3 yield

COROLLARY 1. A connected space X admits a co-H-structure if and only if on each skeleton $X^{(n)}$ there exists such a co-H-structure that the canonical imbedding $X^{(n)} \rightarrow X^{(n+1)}$ is a co-H-map. Furthermore, in this case, the imbedding $X^{(n)} \rightarrow X$ is a co-H-map for suitable co-H-structures on X and $X^{(n)}$.

For a co-H-map $f: X \to Y$ let C_f be its mapping cone and $q: Y \to C_f$ the canonical imbedding. By [3] we may give C_f a co-H-structure in such a way that q is a co-H-map. We now show that the converse of this fact also holds, provided some conditions are satisfied. Recall first that for a co-H-space X with a co-H-structure σ and an inversion map, and any space Y there is a (naturally) split short exact sequence

$$1 \longrightarrow [X, Y \flat Y] \longrightarrow [X, Y \lor Y] \xrightarrow{j_*}_{\underset{\gamma_Y}{\overleftarrow{\leftarrow}}} [X, Y \times Y] \longrightarrow 1,$$

where $j_*\gamma_Y = \text{id}$ with $\gamma_Y([\alpha_1, \alpha_2)]) = [(\alpha_1 \vee \alpha_2)\sigma]$ for $(\alpha_1, \alpha_2) : X \to Y \times Y$. Denote the induced operation on $[X, Y \vee Y]$ additively and let $\beta_Y = \text{id} - \gamma_Y j_*$. Then we get $j_*\beta_Y = 0$.

THEOREM 2. Let $f: X \to Y$ be a map with X and Y 1-connected co-H-spaces. If the mapping cone C_f is a co-H-space, $q: Y \to C_f$ a co-H-map and dim $X < \operatorname{conn} X + \min\{\operatorname{conn} Y, \operatorname{conn} C_f\}$ then $f: X \to Y$ is a co-H-map.

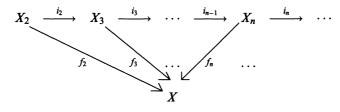
PROOF. Observe that by [3] a co-H-structure on a 1-connected space X admits an inversion. Let $\sigma: Y \to Y \lor Y$ and $\sigma': C_f \to C_f \lor C_f$ be co-H-structures on Y and C_f , respectively and consider the commutative diagram

Then $(qbq)_*\beta_Y\sigma_*([f]) = \beta_{C_f}\sigma'_*q_*([f]) = \beta_{C_f}\sigma'_*([qf]) = 0$. But $\operatorname{conn} q = \operatorname{conn} X$ and by Lemma 1 $\operatorname{conn}(qbq) = \operatorname{conn} q + \min\{\operatorname{conn} Y, \operatorname{conn} C_f\}$, hence $\dim X < \operatorname{conn}(qbq)$. By obstruction theory the map $(qbq)_*$ is an isomorphism, so we get that $\beta_Y\sigma_*[f]) = 0$. From the definition of β_Y we derive that $\sigma f \simeq (f \lor f)\tau$, where $\tau : X \to X \lor X$ is a co-H-structure on Y and finally f is a co-H-map.

In particular, let M(A,n) be the Moore space of type (A,n) for $n \ge 2$. Then dim $M(A,n) \le n+1$ and conn M(A,n) = n-1. Thus we get the following

COROLLARY 2. Let M(A,n) be the Moore space of type (A,n) for $n \ge 2$, X a 2-connected co-H-space and $f: M(A,n) \to X$ a map. If the mapping cone C_f of f is a co-H-space, $q: X \to C_f$ a co-H-map then $f: M(A,n) \to X$ is a co-H-map.

Let now X be a 1-connected space and



its homology decomposition due to Hilton [2, Chapter 8]. Roughly speaking the space X with homology groups $H_2(X)$, $H_3(X)$,... is built-up by a process of successively attaching cones $C(M(H_n(X), n-1))$ to X_{n-1} by the homologically trivial maps $M(H_n(X), n-1) \rightarrow X_{n-1}$ determined by the k'-classes $k'_{n-1} \in [M(H_n(X), n-1), X_n]$, where $M(H_n(X), n-1)$ is the Moore space of type $(H_n(X), n-1)$. Thus, by Lemmas 2 and 3, Theorem 1 and Corollary 2 it may be stated that a dualization of Corollary 5.6 in [5, page 443] there exists.

COROLLARY 3. (1) A 1-connected X space admits a co-H-structure if and only if on each nth stage X_n there exists such a co-H-structure that $i_n: X_n \to X_{n+1}$ is a co-H-map. Furthermore, in this case, $f_n: X_n \to X$ is a co-Hmap for suitable co-H-structures on X and X_n .

(2) If X is a 2-connected co-H-space with homology groups $H_3(X)$, $H_4(X), \ldots$ then the maps $M(H_n(X), n-1) \rightarrow X_{n-1}$ determined by the k'-classes are co-H-maps for all $n \ge 3$.

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