THE FIBER OF THE LINEARIZATION MAP $A(*) \to K(\mathbb{Z})$

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Introduction

There is a linearization map $L\colon * \to \mathbb{Z}$ of functors with smash product (FSPs) inducing the Hurewich map $S^0 \to H\mathbb{Z}$ on the associated ring spectra [Bö, W]. By naturality of the cyclotomic trace map from algebraic K-theory to topological cyclic homology [BHM], there is a commutative square

$$\begin{array}{ccc} A(*) & \xrightarrow{trc} & TC(*) \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle L} \\ K(\mathbb{Z}) & \xrightarrow{trc} & TC(\mathbb{Z}). \end{array}$$

A theorem of B. I. Dundas [Du] asserts that this square becomes homotopy cartesian after p-adic completion, for any prime p. Hence to compute the (p-completed) homotopy fiber of $L\colon A(*) \to K(\mathbb{Z})$ it suffices to determine the homotopy fiber of $L\colon TC(*) \to TC(\mathbb{Z})$. This approach has the advantage that the homotopy types of both these spaces are known, whence only the problem of describing the linearization map between them remains.

The aim of the present paper is to give such a description for odd primes p in the range of homotopy groups where stable homotopy consists only of the image of the J-homomorphism, *i.e.* up to the connectivity of the cokernel of J. The first nontrivial element in the cokernel of J is a class in degree $2p(p-1)-2$. Thus we wish to describe elements in the fiber of L up to degree $2p(p-1)-3$. We call this the "image of J range."

We will prove the following result.

Theorem. *(i) In the image of J range, i.e. through degree $2p(p-1)-3$, the p-primary homotopy groups of the fiber of the linearization map $L\colon A(*) \to K(\mathbb{Z})$ are concentrated in even degrees, and satisfy*

$$\pi_{2n}\,\mathrm{hofib}(L) \cong \begin{cases} \mathbb{F}_p & \text{if } k(p-1) \le n < kp \text{ for some } 2 \le k < p, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) The classes in degrees $2n$ with $n \equiv 0 \bmod p-1$ are the image of a natural map $BSG \to \mathrm{hofib}(L)$, and map to zero in $\pi_{2n}A(*)$.

(iii) The remaining classes, in degrees $2n$ with $k(p-1) < n < kp$, inject into $\pi_{2n}A(*)$, onto direct summands.

Corollary. *There are nontrivial classes in* $K_{2n+1}(\mathbb{Z})$ *with* $n \equiv 0 \mod p-1$ *and* $0 < n < p(p-1)$ *mapping to the classes of order* p *in* $\pi_{2n}\,\mathrm{hofib}(L)$ *under the connecting map* $\Omega K(\mathbb{Z}) \to \mathrm{hofib}(L)$.

There are torsion classes of order p *in* $\pi_{2n-2}\mathcal{P}(*)$ *when* n *satisfies* $k(p-1) \leq n < kp$ *for some* $2 \leq k < p$, *which come from* $\pi_{2n}A(*)$, *map to zero in* $K_{2n}(\mathbb{Z})$, *and are detected in* $\pi_* TC(*)$.

Here $\mathcal{P}(*)$ is the stable pseudoisotopy space of a point.

1. Review of topological cyclic homology

Hereafter let all spectra be implicitly completed at the odd prime p. We will use infinite loop space notation for the connective spectra that appear.

By Theorem 5.17 of [BHM] there is a homotopy cartesian square

$$\begin{array}{ccc} TC(*) & \xrightarrow{\alpha} & Q(\Sigma_+\mathbb{C}P^\infty) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle trf_{S^1}} \\ Q(S^0) & \xrightarrow{1-\Delta_p} & Q_0(S^0) \end{array}$$

Here Δ_p is the pth power map, trf_{S^1} the S^1-transfer for the covering $ES^1 \to BS^1$, and $Q_0(S^0)$ denotes the zero–component of $Q(S^0)$. Similarly we will write $Q_1(S^0) = SG$ for the identity component. Clearly $1 - \Delta_p \simeq *$, so the homotopy type of $TC(*)$ at p is determined by the split fiber sequence

$$\mathrm{hofib}(trf_{S^1}) \to TC(*) \to Q(S^0).$$

The inclusion $S^0 \cong \mathbb{C}P^0_+ \to \mathbb{C}P^\infty$ induces a splitting $Q(\Sigma_+\mathbb{C}P^\infty) \simeq Q(S^1) \times Q(\Sigma\mathbb{C}P^\infty)$. The composite

$$Q(S^1) \to Q(\Sigma_+\mathbb{C}P^\infty) \xrightarrow{trf_{S^1}} Q_0(S^0)$$

is induced by the Hopf map η, and is thus null homotopic at odd p. Therefore trf_{S^1} factors through a restricted transfer map

$$t\colon Q(\Sigma\mathbb{C}P^\infty) \to Q_0(S^0)$$

and $\mathrm{hofib}(trf_{S^1}) \simeq Q(S^1) \times \mathrm{hofib}(t)$. Thus

$$TC(*) \simeq Q(S^0) \times Q(S^1) \times \mathrm{hofib}(t).$$

We remark that η is null at p as an infinite loop map, so t is an infinite loop map, and $\mathrm{hofib}(t)$ is an infinite loop space.

$TC(\mathbb{Z})$ is essentially K-local in the sense that it is the connective cover of its K/p-localization. (This is not quite true. There is a discrepancy in total degree one, but this will not affect our results. We choose to suppress the comments needed to account for this difference.) For brevity we will use the term "K-local" in this modified sense. Recall that K/p-localization is K-localization followed by

p-adic completion, which is implicit in our notation. Hence the linearization map $L\colon TC(*) \to TC(\mathbb{Z})$ induces a map $Q(S^0) \times Q(S^1) \to TC(\mathbb{Z})$ which factors through the K-localizations $L_{K/p}Q(S^0)[0,\infty) = \operatorname{Im} J$ and $L_{K/p}Q(S^1)[0,\infty) = B\operatorname{Im} J$. By [BM, R] the cofiber of the factorized map is SU and there is a splittable fiber sequence

$$\operatorname{Im} J \times B\operatorname{Im} J \to TC(\mathbb{Z}) \to SU$$

identifying $TC(\mathbb{Z})$ as $\operatorname{Im} J \times B\operatorname{Im} J \times SU$.

Hence the linearization map $L\colon TC(*) \to TC(\mathbb{Z})$ is given by the localization maps $e\colon Q(S^0) \to \operatorname{Im} J$ and $Be\colon Q(S^1) \to B\operatorname{Im} J$ on the first two factors, and a map $\operatorname{hofib}(t) \to TC(\mathbb{Z})$ as the third factor. The first two maps are split surjections, with fibers $\operatorname{Cok} J$ and $B\operatorname{Cok} J$, which are $2p(p-1)-3$ and $2p(p-1)-2$- connected, respectively. Thus in the image of J range these maps are equivalences, and we can identify the fiber of L, in this range, with the fiber of the composite map $\ell\colon \operatorname{hofib}(t) \to TC(\mathbb{Z}) \to SU$.

2. The homotopy fiber of the S^1-transfer

To begin this analysis, we first study the map $t\colon Q(\Sigma\mathbb{C}P^\infty) \to Q_0(S^0)$ and its homotopy fiber.

There are natural maps

$$\mathbb{C}P^\infty_+ \to \coprod_{n\geq 0} BU(n) \to \mathbb{Z} \times BU$$

given by inclusion on the $n = 1$ summand, and group completion. Adjunction with respect to the additive infinite loop space structure on the target gives an infinite loop map $\epsilon_+\colon Q(\mathbb{C}P^\infty_+) \to \mathbb{Z} \times BU$. The collapse map $\mathbb{C}P^\infty_+ \to S^0$ is a right inverse to the first map of the cofiber sequence $S^0 \to \mathbb{C}P^\infty_+ \to \mathbb{C}P^\infty$, where the non–base point of S^0 is mapped to the complex line $\mathbb{C}^1 \subset \mathbb{C}^\infty$, viewed as the point $\mathbb{C}P^0$ in $\mathbb{C}P^\infty$. Hence there is a section $Q(\mathbb{C}P^\infty) \to Q(\mathbb{C}P^\infty_+)$, essentially taking a line $L \subset \mathbb{C}^\infty$ to the difference $L - \mathbb{C}^1$.

We view BU as the base point component of $\mathbb{Z} \times BU$. Let $\epsilon\colon Q(\mathbb{C}P^\infty) \to BU$ be an infinite loop map making

$$\begin{array}{ccc} Q(\mathbb{C}P^\infty) & \xrightarrow{\epsilon} & BU \\ \downarrow & & \downarrow \\ Q(\mathbb{C}P^\infty_+) & \xrightarrow{\epsilon+} & \mathbb{Z} \times BU \end{array}$$

homotopy commute. ϵ is well defined up to homotopy. Let $F = \operatorname{hofib}(\epsilon)$ be its homotopy fiber.

The following splitting is due to Segal [Se]. Another proof using transfer maps was given later by Becker [Be]. ((What about James ?))

Proposition (Segal). *The map ϵ has a right homotopy inverse as a map of spaces. Hence there is a homotopy equivalence of spaces*

$$Q(\mathbb{C}P^\infty) \simeq BU \times F.$$

The homotopy groups of F are finite. □

We call F the Becker–Segal space, and ϵ the Becker–Segal map.

In the image of J range we can fiber this splitting over the S^1-transfer map. For this we will use the results of [BS] and [MMM].

Bott periodicity gives an infinite loop space equivalence $\beta\colon \mathbb{Z} \times BU \to \Omega U$. We need to recall the explicit map. On base point components β restricts to a homotopy equivalence $BU \to \Omega SU$, which is the colimit of the Bott maps $\beta_n\colon G_n(\mathbb{C}^{2n}) \to \Omega SU(2n)$ [Bo]. Here $G_n(\mathbb{C}^{n+k})$ denotes the Grassmann manifold of complex n-planes in $\mathbb{C}^{n+k}$. β_n takes an n-plane $X \subset \mathbb{C}^{2n}$ satisfying $X = g(\mathbb{C}^n)$ for some $g \in SU(2n)$ to the loop $t \mapsto [g, \alpha(t)] = g \cdot \alpha(t) \cdot g^{-1} \cdot \alpha(t)^{-1}$ in $SU(2n)$. Here t runs from 0 to 1, and

$$\alpha(t) = \operatorname{diag}(e^{\pi it}, \dots, e^{\pi it}, e^{-\pi it}, \dots, e^{-\pi it})$$

is the diagonal matrix with $e^{\pi it}$ as the first n diagonal entries and $e^{-\pi it}$ as the last n entries. Thus β_n is given in the H-group structure of $\Omega SU(2n)$ as the map rotating by $e^{\pi it}$ on X and by $e^{-\pi it}$ on the orthogonal complement $X^{\perp}$, minus the corresponding map for $X = \mathbb{C}^n$. We speak of β_n as given by conjugation with $\alpha(t)$.

The map $\mathbb{C}P^\infty \to BU$ is the colimit of inclusions $\mathbb{C}P^n = G_1(\mathbb{C}^{n+1}) \to G_n(\mathbb{C}^{2n})$ mapping $L \to X = L \oplus \mathbb{C}^{n-1}$. Here $\mathbb{C}^{n+1} \subset \mathbb{C}^{2n}$ is viewed as the span of the first and n last basis vectors $e_1, e_{n+1}, \dots, e_{2n}$, and $\mathbb{C}^{n-1}$ is its orthogonal complement. Thus the composite $\mathbb{C}P^n \to G_n(\mathbb{C}^{2n}) \xrightarrow{\beta_n} \Omega SU(2n)$ maps to loops of matrices acting trivially upon the summand $\mathbb{C}^{n-1}$, and thus factors through $\Omega SU(n+1)$. The map $\mathbb{C}P^n \to \Omega SU(n+1)$ is given by conjugation with

$$\alpha'(t) = \operatorname{diag}(e^{\pi it}, e^{-\pi it}, \dots, e^{-\pi it}).$$

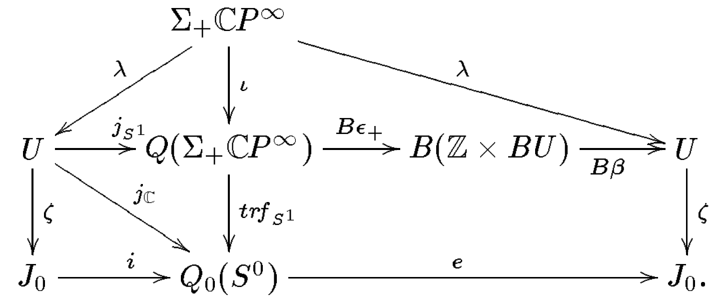
Since the diagonal matrix with $e^{\pi it}$ in every diagonal entry is in the center of $U(n+1)$, this map is also given by conjugation with

$$\alpha''(t) = \operatorname{diag}(e^{2\pi it}, 1, \dots, 1).$$

In the limit we find that $\mathbb{C}P^\infty \to \Omega SU$ is given in the H-group structure on the target as the map taking a line $L \subset \mathbb{C}P^\infty$ to the loop of rotations around L, leaving $L^\perp$ fixed, minus the loop of rotations around $\mathbb{C}^1$, leaving $(\mathbb{C}^1)^\perp$ fixed.

Define the map $\lambda\colon \Sigma_+\mathbb{C}P^\infty \to U$ by taking a point $(L, z) \in \Sigma_+\mathbb{C}P^\infty = \mathbb{C}P^\infty_+ \wedge S^1$ to the unitary matrix rotating by z on L and leaving the orthogonal complement $L^\perp$ fixed. This is the map denoted $\lambda_{\mathbb{C}}$ in [MMM]. The discussion above shows that $\beta \circ \epsilon\colon \mathbb{C}P^\infty \to \Omega SU$ is the unique lift to SU of the difference of the adjoint of λ and the constant map $\mathbb{C}P^\infty \to \mathbb{C}P^0 \to \Omega U$ rotating about $\mathbb{C}^1$. This is just the constant map $\mathbb{C}P^\infty \to \mathbb{C}P^0 \to \mathbb{C}P^\infty$ followed by the adjoint of λ.

Consider the following diagram.



Proposition. *The diagram above homotopy commutes. The bottom right square is homotopy commutative as a square of infinite loop spaces and maps.*

Proof. We describe the notation while explaining why the diagram commutes.

ϵ_+, β and λ were defined above. B denotes a delooping. ι is the adjunction unit. The top right triangle commutes when restricted to $S^1 = \Sigma_+\mathbb{C}P^0 \subset \Sigma_+\mathbb{C}P^\infty$, essentially because all the maps involved induce π_1-isomorphisms.

Subtracting off this summand, we can compare two maps $\Sigma\mathbb{C}P^\infty \to U$ which lift uniquely over SU. These are adjoint to the two maps shown to be homotopic by the discussion preceding the diagram. Thus the top triangle homotopy commutes.

Let $g \in \mathbb{Z}_p^\times \cong \mathbb{Z}/(p-1)\times\mathbb{Z}_p$ be a topological generator, *e.g.* an integer generating the units of $\mathbb{Z}/p^2$. Then $\operatorname{Im} J$ is defined as the homotopy fiber of $\psi^g - 1\colon \mathbb{Z}\times BU \to BU$, and J_0 is the zero component of $\operatorname{Im} J$. Here ψ^g is the Adams operation. ζ is the induced connecting map in the corresponding Puppe fiber sequence. e is the Adams e-invariant, which by definition is an equivalence in the image of J range.

i is a section to e. $j_{\mathbb{C}}$ is the complex J-homomorphism $U \to SG = Q_1(S^0)$ shifted into the zero–component. Hence the bottom left triangle homotopy commutes. j_{S^1} is the S^1-equivariant J-homomorphism, taking a unitary isometry $\mathbb{C}^n \to \mathbb{C}^n$ to the stable class of the map of unit spheres $S^{2n-1} \to S^{2n-1}$, viewed as a map of free S^1-spaces. The target space is identified with $Q(\Sigma_+\mathbb{C}P^\infty)$ via the Segal–tom Dieck splitting, as explained in [BS]. The S^1-transfer map trf_{S^1} forgets the free S^1-action. Hence the inner triangle commutes. The top left triangle homotopy commutes by Theorem B of [MMM].

We claim that the bottom right square homotopy commutes as a square of infinite loop space maps. This is equivalent to the assertion that the two maps $\Sigma_+\mathbb{C}P^\infty \to J_0$ given by precomposing with ι are homotopic as maps of spaces. But this is clear from the perimeter of the diagram, since $e \circ i$ is homotopic to the identity. $\square$

We now combine the two infinite loop space squares.

$$\begin{array}{ccccc}
Q(\Sigma\mathbb{C}P^\infty) & \longrightarrow & Q(\Sigma_+\mathbb{C}P^\infty) & \xrightarrow{trf_{S^1}} & Q_0(S^0) \\
\big\downarrow{\scriptstyle B\epsilon} & & \big\downarrow{\scriptstyle B\epsilon_+} & & \big\downarrow{\scriptstyle e} \\
SU & \longrightarrow & U & \xrightarrow{\zeta} & J_0
\end{array}$$

t is the composite across the top, while we write ζ again for the bottom composite. Let JSU be the homotopy fiber of $\psi^g - 1\colon BSU \to BSU$. Then $JSU \simeq J_0$ since p is odd, so we can identify the homotopy fiber of ζ with SU again. Hence there is a natural map of homotopy fibers $B\epsilon' \colon \operatorname{hofib}(t) \to SU$. Let BF' be the homotopy fiber of $B\epsilon'$. (We write these spaces and maps as deloopings to make the notation more consistent.)

Then there is a 3×3 diagram of fiber sequences

$$
\begin{array}{ccccc}
BF' & \longrightarrow & \operatorname{hofib}(t) & \xrightarrow{B\epsilon'} & SU \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \Omega(\psi^g - 1)} \\
BF & \longrightarrow & Q(\Sigma\mathbb{C}P^\infty) & \xrightarrow{B\epsilon} & SU \\
\downarrow & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \zeta} \\
\operatorname{Cok} J & \longrightarrow & Q_0(S^0) & \xrightarrow{e} & J_0
\end{array}
$$

Hence $F' = \Omega BF'$ has finite homotopy groups, and agrees with F in the image of J range. $\operatorname{Cok} J$ is by definition the homotopy fiber of e.

Proposition. *The delooped Becker–Segal map $B\epsilon$ lifts to a map of infinite loop spaces $B\epsilon' \colon \operatorname{hofib}(t) \to SU$. In the image of J range $B\epsilon'$ admits a right homotopy inverse as a map of spaces, and its homotopy fiber can be identified with BF. Hence there is a homotopy equivalence of spaces*

$$\operatorname{hofib}(t) \simeq SU \times BF$$

in the image of J range.

Proof. Choose sections $j \colon SU \to Q(\Sigma\mathbb{C}P^\infty)$ and $i \colon J_0 \to Q_0(S^0)$ to $B\epsilon$ and e respectively, in the homotopy category of spaces. Then the left square of

$$
\begin{array}{ccccc}
SU & \xrightarrow{j} & Q(\Sigma\mathbb{C}P^\infty) & \xrightarrow{B\epsilon} & SU \\
\downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \zeta} \\
J_0 & \xrightarrow{i} & Q_0(S^0) & \xrightarrow{e} & J_0
\end{array}
$$

will homotopy commute in the image of J range. Hence there is a map of fibers $SU \to \operatorname{hofib}(t)$ defined in the image of J range, whose composite with ϵ' to SU is an equivalence in the same range. □

((Can j be chosen as j_{S^1} ?))

3. Homotopy of the Becker–Segal space

Let S^0/p be the mod p Moore spectrum. There is an Adams map $\Sigma^{2p-2}S^0/p = S^{2p-2}/p \to S^0/p$ inducing an isomorphism $\pi_*(X;\mathbb{F}_p) \to \pi_{*+2p-2}(X;\mathbb{F}_p)$ when X is a K-local infinite loop space. As is usual, this map is denoted v_1, and we note that the mod p homotopy of our K-local infinite loop spaces are free modules over the polynomial algebra $\mathbb{F}_p[v_1]$.

Proposition. *In the image of J range, $\pi_*(F)$ is a nontrivial cyclic p-group in odd degrees $*$ satisfying $2k(p-1)+1 \le * \le 2kp-3$ for some $2 \le k < p$, and trivial otherwise.*

Proof. The S^1-equivariant sphere spectrum, with underlying S^1-space $Q_{S^1}(S^0)$, is a split spectrum in the sense of [LMS]. Hence so is its K-localization $\operatorname{Im} J$. Thus

$$\operatorname{Im} J^*_{S^1}(ES^1_+) \cong \operatorname{Im} J^*(\mathbb{C}P^\infty_+) \quad \text{and} \quad \operatorname{Im} J^{S^1}_*(ES^1_+) \cong \operatorname{Im} J_*(\mathbb{C}P^\infty_+).$$

In particular we can compute $\pi_* Q(\mathbb{C}P^\infty_+)$ (resp. $\pi_* Q(\mathbb{C}P^\infty)$) in the image of J range by considering truncated versions of the Tate spectral sequence for the Tate construction $\hat{\mathbb{H}}(S^1, \operatorname{Im} J)$ of [BM]. This Tate construction is denoted $t_{S^1}(\operatorname{Im} J)^{S^1}$ in the notation of [GM].

Similar remarks apply in mod p homotopy. Hence there is a first quadrant Atiyah–Hirzebruch spectral sequence

$$E^2_{s,*} = \tilde{H}_s(\mathbb{C}P^\infty; \pi_*(\operatorname{Im} J; \mathbb{F}_p)) \Longrightarrow \pi_{s+t}(\operatorname{Im} J \wedge \mathbb{C}P^\infty; \mathbb{F}_p)$$

with $\pi_*(\operatorname{Im} J; \mathbb{F}_p) \cong \mathbb{F}_p[a,b]/(a^2=0)$, $\deg a = 2p-3$, $\deg b = 2p-2$ and mod p Bockstein $\beta_1(b) = a$, mapping into the upper half plane Tate spectral sequence

$$\begin{aligned}\hat{E}^2_{s,*} &= \hat{H}^{-s}(S^1; \pi_*(\operatorname{Im} J; \mathbb{F}_p)) \\ &\cong \mathbb{F}_p[t, t^{-1}, a, b]/(a^2 = 0)\end{aligned}$$

with a bidegree shift of $(2,0)$. The point of making this comparison is that the latter spectral sequence is an algebra spectral sequence. The periodic element $t \in \hat{H}^2(S^1; \mathbb{F}_p)$ sits in bidegree $(-2,0)$ in the Tate spectral sequence. There are generators $x_i \in \tilde{H}_{2i}(\mathbb{C}P^\infty; \mathbb{F}_p)$ in bidegree $(2i,0)$ of the Atiyah–Hirzebruch spectral sequence, mapping to t^{-i-1}, for all $i \ge 1$.

The first nontrivial differentials in the Tate spectral sequence are d^{2p-2}-differentials corresponding to the first Steenrod pth power operation P^1 appearing as the first nontrivial k-invariant of $Q(S^0)$ and $\operatorname{Im} J$ at p. Since $P^1(t) = t^p$ there is a nonzero differential $d^{2p-2}(t) = t^p \cdot a$. (We may define the generator a by this identity, and then choose b to satisfy $\beta_1(b) = a$.)

We may also truncate the Tate spectral sequence to the second quadrant, to compute $\operatorname{Im} J^*(\mathbb{C}P^\infty_+)$. Since $Q(S^0)$ splits off from $Q(\mathbb{C}P^\infty_+)$, the zeroth column (= the vertical axis) consists of infinite cycles. Hence b is an infinite cycle in the Tate spectral sequence. It is not a d^2-boundary, for $t^{-1}a$ is an integral class, and so $d^2(t^{-1}a) = 0$ by comparison with the corresponding spectral sequence in integral homotopy. Thus b is a permanent cycle in the Tate spectral sequence, for bidegree reasons. In particular b acts on the entire Tate spectral sequence, and thus also on the Atiyah–Hirzebruch spectral sequence, representing the v_1-action on mod p homotopy.

We find $d^{2p-2}(t^i \cdot b^j) = it^{i+p-1} \cdot ab^j$ for all i, j. This leaves the E^{2p-1}-term

$$\hat{E}^{2p-1}_{*,*} = \mathbb{F}_p[t^p, t^{-p}, b]\{1, t^{-1}a\}$$

and the next possible nonzero differentials begin with $d^{2(p^2-1)}(t^p)$, for bidegree reasons. As is shown in [BM], these differentials are indeed nonzero, but we shall shortly see that these second and later generations of differentials do not play a part within the image of J range.

By naturality, these differentials from the Tate spectral sequence translate over to the Atiyah–Hirzebruch spectral sequence, yielding the following E^{2p-1}-term:

$$E^{2p-1}_{*,*} = \mathbb{F}_p[b]\{x_1, \dots, x_{p-1}, x_p a, x_{2p-1}, x_{2p} a, x_{3p-1}, x_{3p} a, \dots\}$$

which in the image of J range is also the E^∞-term. For a first–quadrant $d^{2(p^2-1)}$-differential cannot affect classes in total degree below $2(p^2-1)-1 > 2p(p-1)-2$.

In view of the splitting $Q(\mathbb{C}P^\infty) \simeq BU \times F$, the only classes in degrees $2, \dots, 2p-2$, namely $x_1, \dots, x_{p-1}$, must represent the $p-1$ generators of $\pi_*(BU; \mathbb{F}_p)$ as a (free) $\mathbb{F}_p[v_1]$-module, corresponding to the $p-1$ summands of the Adams splitting of BU. By naturality of the v_1-action on mod p homotopy, any homotopy section $BU \to Q(\mathbb{C}P^\infty)$ maps $\pi_*(BU; \mathbb{F}_p)$ precisely onto the summands $\mathbb{F}_p[b]\{x_1, \dots, x_{p-1}\}$ of the above E^∞-term, *i.e.* the classes in filtration $2p-2$ or less. Consequently the remaining classes represent $\pi_*(F; \mathbb{F}_p)$, in pairs linked by (higher) Bockstein operations, since $\pi_*(F)$ is all torsion. We have

$$\pi_*(F; \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p & \text{if } 2k(p-1) < * < 2kp-1 \text{ for some } 1 < k < p, \\ 0 & \text{otherwise} \end{cases}$$

in the image of J range. For every total degree there are permanent cycles in at most one bidegree, in this range, whence the integral homotopy groups of F are finite cyclic, and located in the "lower half" of these total degrees. This is immediate from the universal coefficient theorem for mod p homotopy. The proposition follows. □

The order of these cyclic groups is determined by the following theorem of K. Knapp [Kn].

Theorem (Knapp). *The order of* $(\mathrm{Im}\, J)_*(\mathbb{C}P^\infty)$ *in degree* $* = 2n-1$ *has p-adic valuation*

$$v_p(|(\mathrm{Im}\, J)_{2n-1}(\mathbb{C}P^\infty)|) = \sum_{i=1}^{[\frac{n-1}{p-1}]} (1 + v_p(i)) \quad - v_p(n!).$$

When $n \le p(p-1)$ *this equals* $[\frac{n-1}{p-1}] - [\frac{n}{p}]$, *which in turn equals* 0 *or* 1. □

We summarize.

Corollary. *In the image of* J *range,* $\pi_*(F) \cong \mathbb{F}_p$ *in odd degrees* $*$ *satisfying* $2k(p-1)+1 \le * \le 2kp-3$ *for some* $2 \le k < p$, *and* $\pi_*(F) = 0$ *otherwise.* □

4. The linearization map

Finally we look at the homotopy fiber of the linearization map $A(*) \to K(\mathbb{Z})$. By the cited theorem of Dundas, this is equivalent to the homotopy fiber of the linearization map $TC(*) \to TC(\mathbb{Z})$. In the image of J range we have noted that this in turn agrees with the homotopy fiber of the rational equivalence $\ell\colon \mathrm{hofib}(t) \to SU$.

Proposition. *There is a $(4p-6)$-connected map $BSG \to \operatorname{hofib}(A(*) \to K(\mathbb{Z}))$.*

Proof. Recall that $SG = Q_1(S^0)$ is $(2p-4)$-connected. So there is a $(4p-6)$-cartesian square

$$\begin{array}{ccc} A(BSG) & \longrightarrow & A(*) \\ \downarrow & & \downarrow{\scriptstyle L} \\ A(*) & \xrightarrow{L} & K(\mathbb{Z}). \end{array}$$

Compare Theorem 9.10 of [BM]. This uses that $A(X)$ is a 1-analytic functor in the sense of Goodwillie [Go]. Furthermore the derivative of $A(X)$ at $X = *$ is the sphere spectrum, *i.e.* $Q(BSG) \to \operatorname{hofib}(A(BSG) \to A(*))$ is a $(4p-6)$-connected map. Finally $BSG \to Q(BSG)$ is also $(4p-6)$-connected. The result follows. □

These observations may be summarized in the diagram

$$\begin{array}{ccccc} BSG & \longrightarrow & \operatorname{hofib}(\ell) & & \\ & & \downarrow & & \\ BF' & \longrightarrow & \operatorname{hofib}(t) & \xrightarrow{B\epsilon'} & SU \\ & & \downarrow{\scriptstyle \ell} & & \\ & & SU & & \end{array}$$

where $BSG \to \operatorname{hofib}(\ell)$ is at least $(4p-6)$-connected.

((What is the map $BF' \to SU$? Is some operation $SU \to SU$ involved ?))

In the image of J range we get an exact sequence

$$0 \longrightarrow \pi_{2n+1}SU \xrightarrow{\ell} \pi_{2n+1}SU \longrightarrow \pi_{2n}\operatorname{hofib}(\ell) \longrightarrow \pi_{2n}BF' \longrightarrow 0.$$

Furthermore $\pi_{2n+1}\operatorname{hofib}(\ell) = 0$. Below degree $4p-6$ the group $\pi_{2n}BF'$ vanishes, and the rational equivalence (hence injection) $\pi_{2n+1}SU \to \pi_{2n+1}SU$ has cokernel $\pi_{2n}BSG$, which is trivial unless $n = p-1$. In this case the cokernel is $\mathbb{F}_p$.

ℓ induces a homomorphism of $\mathbb{F}_p[v_1]$-modules $\pi_*(\operatorname{hofib}(t); \mathbb{F}_p) \to \pi_*(SU; \mathbb{F}_p)$. The target is a free module of rank $p-1$ with generators in degrees $3, 5, \dots, 2p-1$. The map induces an isomorphism onto the $p-2$ first of these generators, in the degrees $3, 5, \dots, 2p-3$, and is zero in degree $2p-1$. This follows because the fiber agrees with BSG in this range. Hence by the v_1-action, ℓ induces an isomorphism $\pi_{2n+1}\operatorname{hofib}(t) \to \pi_{2n+1}SU$ for all $n \not\equiv 0 \bmod p-1$. So $\pi_{2n}\operatorname{hofib}(\ell) \cong \pi_{2n}BF'$ in these degrees.

$$\begin{array}{ccccc} \pi_{2n+1}(BF'; \mathbb{F}_p) & \longrightarrow & \pi_{2n+1}(\operatorname{hofib}(t); \mathbb{F}_p) & \xrightarrow{B\epsilon'} & \pi_{2n+1}(SU; \mathbb{F}_p) \\ & & \downarrow{\scriptstyle \ell} & & \\ & & \pi_{2n+1}(SU; \mathbb{F}_p) & & \\ & & \downarrow & & \\ \pi_{2n}(BSG; \mathbb{F}_p) & \longrightarrow & \pi_{2n}(\operatorname{hofib}(\ell); \mathbb{F}_p) & & \end{array}$$

Hereafter suppose that $n \equiv 0 \mod p-1$. In the image of J range $\pi_{2n}BF' = \pi_{2n}BF = 0$ in these degrees. Hence $\pi_{2n+1}(\mathrm{hofib}(t);\mathbb{F}_p) \cong \pi_{2n+1}(SU;\mathbb{F}_p)$, and $\pi_{2n}(\mathrm{hofib}(t);\mathbb{F}_p) = 0$. Thus ℓ induces a homomorphism of free rank one $\mathbb{F}_p[v_1]$-modules, which takes the generator to zero. Thus this homomorphism is zero, and there is an isomorphism $\pi_{2n+1}(SU;\mathbb{F}_p) \to \pi_{2n}(\mathrm{hofib}(\ell);\mathbb{F}_p)$, again of free rank one $\mathbb{F}_p[v_1]$-modules.

Thus $\pi_{2n}(BSG;\mathbb{F}_p) \to \pi_{2n}(\mathrm{hofib}(\ell);\mathbb{F}_p)$ is a map of free rank one $\mathbb{F}_p[v_1]$-modules with generators in degree $2p-2$, which takes a module generator to a module generator, and hence is an isomorphism (in the image of J range). Thus $\mathbb{F}_p \cong \pi_{2n}BSG \to \pi_{2n}\,\mathrm{hofib}(\ell)$ is a surjection onto a nontrivial group, *i.e.* $\pi_{2n}\,\mathrm{hofib}(\ell) \cong \mathbb{F}_p$ in all these cases. (The first element of order p^2 in π_*BSG is in degree $2p(p-1)$, just outside the image of J range.)

Our main result follows.

Theorem. *(i) In the image of J range, i.e. through degree $2p(p-1)-3$, the p-primary homotopy groups of the fiber of the linearization map $L\colon A(*) \to K(\mathbb{Z})$ are concentrated in even degrees, and satisfy*

$$\pi_{2n}\,\mathrm{hofib}(L) \cong \begin{cases} \mathbb{F}_p & \text{if } k(p-1) \le n < kp \text{ for some } 2 \le k < p \\ 0 & \text{otherwise.} \end{cases}$$

(ii) The classes in degrees $2n$ with $n \equiv 0 \mod p-1$ are the image of the natural map $BSG \to \mathrm{hofib}(L)$, and map to zero in $\pi_{2n}A(*)$.

(iii) The remaining classes, in degrees $2n$ with $k(p-1) < n < kp$, inject into $\pi_{2n}A(*)$, onto direct summands.

Proof. We have proved claim (i) above. The second part of claim (ii) follows since the natural map $BSG \to A(BSG) \to A(*)$ factors through $*$. Claim (iii) is clear since the remaining torsion classes map to the direct summand π_*BF' in $\pi_*TC(*)$, by a map factoring through $A(*)$. □

References

[Be] J. C. Becker, *Characteristic classes and K-theory*, Algebraic and Geometrical Methods in Topology, Lecture notes in math., vol. 428, Springer Verlag, 1974, pp. 132–143.

[BS] J. C. Becker and R. E. Schultz, *Equivariant function spaces and stable homotopy theory*, Comment. Math. Helv. **49** (1974), 1–34.

[Bo] R. Bott, *The stable homotopy of classical groups*, Ann. of Math. **70** (1959), 313–337.

[Bö] M. Bökstedt, *Topological Hochschild homology*, Bielefeld preprint.

[BHM] M. Bökstedt, W. C. Hsiang and I. Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, Invent. Math. **11** (1993), 465–540.

[BM] M. Bökstedt and I. Madsen, *Topological cyclic homology of the integers*, Asterisque **226** (1994), 57–143.

[Du] B. I. Dundas, *Relative K-theory and topological cyclic homology*, Aarhus preprint (1995).

[Go] T. Goodwillie, *Calculus II. Analytic functors*, K-Theory **5** (1992), 295–332.

[GM] J. P. C. Greenlees and J. P. May, *Generalized Tate cohomology*, Mem. Amer. Math. Soc. **543** (1995).

[Kn] K. Knapp, *On the bi–stable J-homomorphism*, Algebraic topology, Aarhus, Lecture notes in Math., vol. 763, Springer Verlag, 1978, pp. 13–22.

[LMS] L. G. Lewis, Jr., J. P. May and M. Steinberger, *Equivariant Stable Homotopy Theory*, Lecture notes in Math., vol. 1213, Springer Verlag, 1986.

[MMM] B. M. Mann, E. Y. Miller and H. R. Miller, *S^1-equivariant function spaces and characteristic classes*, Trans. Amer. Math. Soc. **295** (1986), 233–256.

[R] J. Rognes, *Characterizing connected K-theory by homotopy groups*, Math. Proc. Camb. Phil. Soc **114** (1993), 99–102.

[Se] G. Segal, *The stable homotopy of complex projective space*, Quart. Jour. of Math. **24** (1973), 1–5.

[W] F. Waldhausen, *Algebraic K-theory of topological spaces. I*, Algebraic and Geometric Topology, Proc. of Symp. in Pure Math, vol. 32, Amer. Math. Soc., 1978, pp. 35–60.

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