A "Sullivan Conjecture" for Equivariant Structure Sets

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For several decades the problem of classifying G-manifolds up to equivariant homeomorphism or diffeomorphism was approached by means of equivariant surgery which necessitated ~~a~~ a "gap hypothesis" essentially that dimensions of fixed point sets of all subgroups, be less than half the dimension of any larger stratum they lie in. In this note we sketch a new approach to this problem in the case $G = \mathbb{Z}/p$, p an odd prime under the condition that $\dim F < \dim M - 3$, where M is the manifold acted on and F is the fixed point set. Our results are strongly motivated by the earlier works [CW2][We2][Ch]; they readily imply the main result of [CW1] on $\mathbb{Z}/p$ actions on the plane; Elsewhere we hope to do some calculations with the theoretical machinery developed here.

We suppose that M is a $\mathbb{Z}_p$-manifold. For simplicity of notation we will assume F is connected. By $S^{\mathbb{Z}_p}(M)$ we mean $\{(M', f) \mid M'$ is a $\mathbb{Z}_p$-manifold and $f : M' \to M$ is an equivariant homotopy equivalence$\}$ / equiv. homeomorphism

Let $\widetilde{Emb}_i(F \hookrightarrow M)$ denote the Δ-set of blocked embeddings e of F in M with a homotopy of e to the inclusion i of F as the fixed set of $\mathbb{Z}_p$ acting on M. Note that $\mathbb{Z}_p$ acts on $\widetilde{Emb}_i(F \subset M)$.

Using the embedding theorem of Browder, Casson, Haefliger, Sullivan and Wall (see e.g. [Wa, We 1]) there is a map

$$\widetilde{S}^{\mathbb{Z}_p}(M) \longrightarrow \left(\widetilde{Emb}_i(F \subset M)\right)^{h\mathbb{Z}_p}$$

There is a restriction map of these sets to $TRep(\mathbb{Z}_p)$ the "topological representations of $\mathbb{Z}_p$" by considering the germ of an action near a fixed point. Let ρ be such a representation. $\widetilde{S}^{\mathbb{Z}_p}_{\rho}(M)$ is the analogous object where we assume that the normal representation at the fixed set is given by ρ. One can similarly consider $\left(\widetilde{\mathrm{Emb}}_i(F \subset M)\right)^{h\mathbb{Z}_p}_{\rho}$ by insisting our embeddings are standardized and fixed near a point and demanding that all ~~[illegible]~~ the pseudoisotopy data be trivial in a germ near the fixed set (i.e. be given exactly by the data of being genuinely fixed by our initial $\mathbb{Z}_p$ action.)

Our result is the following

__Theorem__ If p is an odd prime, there is an isomorphism

$$S^{\mathbb{Z}_p}_{\rho}(M) \cong S^{\mathbb{Z}_p,\mathrm{iso}}_{\rho}(M) \times \left(\widetilde{\mathrm{Emb}}_i(F \subset M)\right)^{h\mathbb{Z}_p}_{\rho}$$

Note that $S^{\mathbb{Z}_p,\mathrm{iso}}_{\rho}(M)$ can be computed by stratified surgery (see e.g. [We 1]).

As a first, inefficient, pass at calculation — just sufficient to show that one doesn't need an infinite amount of homotopy data, we point out the following.

__Proposition:__ $S^{\mathbb{Z}_p}(M) \rightarrowtail S^{\mathbb{Z}_p,\mathrm{iso}}(M) \times \widetilde{\mathrm{Emb}}_i(F \times L^k_p \hookrightarrow M \times_{\mathbb{Z}_p} S^k)$ for $k > \dim M$

This latter set of embeddings can be analyzed, in principle, by the techniques of [GKW]. Ideally, our theorem should combine with these embedding theory techniques to give a "calculus of symmetry" ~~contexts~~. We note that one can readily generalize our theorem to semi-free actions of odd order groups. Presumably there are more complicated stratified versions for larger groups, but we have not yet pursued these.

The theorem is proven by ~~constructing an~~ observing that there is an "assembly" construction

$\widetilde{Emb}_i(F\subset M)^{h\mathbb{Z}_p} \longrightarrow \widetilde{IS}(M)$ where $\widetilde{IS}$ is the Δ-set of $\mathbb{Z}_p$-isovariant Poincaré complexes with an equivariant homotopy equivalence to M. After all, $\widetilde{Emb}_i(F\subset M)^{h\mathbb{Z}_p}$ can be thought of a space of fibrations over $B\mathbb{Z}_p$, whose total space includes a complement whose boundary is given as an $S(\nu)$

fibration over $F \times B\mathbb{Z}_p$. Ignoring the map to $B\mathbb{Z}_p$, or better yet, using it to define the p-fold cover and it's $\mathbb{Z}_p$-action one sees the element of $\widetilde{\mathcal{I}s}(M)_p$.

Note that such a Poincaré object is automatically isovariantly finite because it is equivariantly, and finiteness can be measured by equivariant chain ~~constructions~~ complexes. (The ∂ is taken care of by "p") (Reference to Bagglivio, I think...)

Note that since $\widetilde{Emb}_i(F \subset M) \to \widetilde{PE}_i(F \subset M)$ is a homotopy equivalence this map is a homotopy equivalence.

Finally there is the issue of manifold realization. This follows from an argument of [We 2] relying on [CWY]. As we have an isovariant Poincaré complex equivariantly h.e. to M after crossing with $(\mathbb{C}P^2)^p$ (with permutation action) a number of times one can see that the action is realized stably by $M \times (\mathbb{C}P^2)^{p \cdot r}$. Since $(\mathbb{C}P^2)^p$ is a periodicity manifold [Y], if a Poincaré complex has realization after taking product with this space, it does so immediately. ~~This completes~~

This completes our sketch of the theorem. The succeeding proposition is similar except that one notes that the homotopy type of $X/\mathbb{Z}_p$ for a d-dimensional free $\mathbb{Z}_p$-space can be obtained from the $d+1$ type of $(X \times S^m)/\mathbb{Z}_p$ for $m \geq d+1$.

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